Real Space Renormalization Group for Classical Systems

-A Pedagogical Approach

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Abstract

We apply the real space Renormalisation Group (RNG) technique to a variety of one-dimensional Ising chains. We begin by recapitulating the work of Nauenberg for an ordered Ising chain, namely the decimation approach. We extend this work to certain non-trivial situation namely, the Alternate Ising Chain and Fibonacci Ising chain. Our approach is pedagogical and accessible to undergraduate students who have had a first course in statistical mechanics.

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Contents

List of Figures

Introduction

There is a deep and useful connection between Statistical Mechanics and Quantum Field Theory. Kenneth Wilson appreciated this connection and applied the renormalization ideas to statistical mechanics[1]. Application of these techniques to both classical and quantum many body problems have seen success. However, RNG calculations are often very complex and the approximations made are sometimes obscure. Often, one has to resort to extensive numerical calculations.

The present work is written in the spirit of conveying some essential ideas of RNG to a beginner and applying this approach to more complicated Ising chains. We present some pedagogical examples of a form of real space RNG termed Decimation. This technique was introduced by Michael Nauenberg in the context of the one-dimensional Ising model. Unfortunately, this attractive piece of work[2] is marked by several typographical errors. We present Nauenberg's work in a simplified (and hopefully error-free) fashion. We extend it to related Hamiltonians such as the Alternate Ising model and Fibonacci chain Ising model.

The RNG strategy can be symbolically stated as follows. It transforms the Hamiltonian, e.g. $H^{'} = \mathbb{R}(H)$. Next, one iterates it, $H^{''} = \mathbb{R}(H')$ until one obtains a fixed point Hamiltonian, $H^* = \mathbb{R}(H^*)$. The flow towards the fixed point Hamiltonian and the Hamiltonian H^* itself yields insight into the physical properties of the system. Wilson suggested such a procedure and was able to elicit the critical properties of the 2D and the 3D Ising model and a famous quantum system namely, the Kondo problem[3].

In Sec.(2), we recapitulate the work of Nauenberg and describe how decimation is carried out for the one-dimensional Ising model. In Sec. 3, we extend this approach to alternate Ising model where the coupling is alternate like in a binary alloy. In Sec. 4, we discuss the Fibonacci Ising chain. Sec. 5 constitutes the conclusion.

One Dimensional Ising Model

We start with the familiar one-dimensional Ising model for N spins, $S_i = \pm 1$, $i = 1, 2...N$, with nearest neighbour coupling constant J, see Fig.(1).
The Hamiltonian H_N for this model is written as,

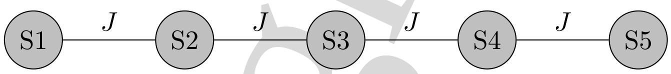

Figure 1: One Dimensional Ising Spin model

$$H_N = -\frac{J}{kT}\sum_{i=1}^{N} S_i S_{i+1},$$

where $S_{N+1} = S_1$, J is the nearest neighbiour exchange coupling, T is the temperature, and k is the Boltzmann constant. We divide the coupling constant by kT for the sake of convenience in further derivations. One can consider the dimensionless Hamiltonian H_N, without loss of generality,

$$H_N(K) = -K\sum_{i=1}^{N} S_i S_{i+1} \qquad (\frac{J}{kT} = K)$$

Note that $K > 0$ implies ferromagnetism.

Decimation

Let $\mathbb{P}$ be the transfer matrix such that $\mathbb{P}(i, i+1) = \exp(KS_iS_{i+1})$. Thus, the canonical partition function Z_N is given by,

$$\begin{aligned} Z_N &= \sum_{s_1,s_2,s_3...} exp(-H_N(K)) = \sum \mathbb{P}(S_1S_2)\mathbb{P}(S_2S_3)\mathbb{P}(S_3S_4)... \\ \mathbb{P} &= \begin{bmatrix} e^K & e^{-K} \\ e^{-K} & e^K \end{bmatrix} \end{aligned}$$

As the elements of the matrix depend on the product S_iS_{i+1} which is same for all i, we can write

$$Z_N = \sum_{s_1,s_2,s_3...} (\mathbb{P}(\mathbb{K})^N) = Tr(\mathbb{P}(\mathbb{K})^N)$$

Now, instead of computing the usual partition sum as shown above, we consider only the partial sum of $exp[-H_N(K)]$ over all possible values of even spins, $S_i = \pm1, i = 2, 4, ...$ and for even N we obtain a scaled partition function $exp[-H_N(K')]$, (see Fig. (2)).

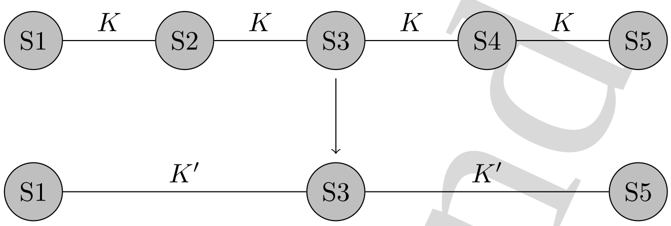

Figure 2: Decimation

The Hamiltonian becomes,

$$\sum_{[s_2s_4..s_N]} exp[-H_N(K)] = \mathbb{P}^2_{S_1S_3}\mathbb{P}^2_{S_3S_5}....\mathbb{P}^2_{S_{N-1}S_1}$$

The idea behind this partial summation is to find a renormalization transformation K$\rightarrow K^{'}$ such that,

$$\begin{aligned} \mathbb{P}^2(K) &= exp[2g(K)]\mathbb{P}(K^{'}) \\ \sum exp(-H_N(K)) &= Tr\mathbb{P}(\mathbb{K})^N = Tr[\mathbb{P}(\mathbb{K})^2]^{\frac{N}{2}}, \end{aligned} \tag{1}$$

where g(K) is a scalar function of K. Then $K^{'}$ can be interpreted as an effective Ising coupling constant for the remaining odd spins $S_i, i = 1, 3, 5....N-1$ and Eq. (1) may (formally) be written as.

$$\begin{aligned} exp(-H_N(K)) &= \mathbb{P}(\mathbb{K})^N \\ &= [exp(2g(K))\mathbb{P}(\mathbb{K}')]^{N/2} \\ &= exp(Ng(K))[(\mathbb{P}(K'))]^{N/2} \\ &= exp(Ng(K))exp(-H_{\frac{N}{2}}(K')) \end{aligned}$$

Thus, the resulting equation becomes,

$$\sum_{[s_1s_2..s_N]} exp[-H_N(K)] = \sum_{[s_1s_3..s_N]} exp[-H_{N/2}(K^{'}) + Ng(K)] \tag{2}$$

To make the procedure clear we discuss the case of 3 spins,

$$\begin{aligned} e^{K'S_1S_3} * e^{2g(K)} &= \sum_{S_2=+1}^{-1} e^{KS_1S_2} * e^{KS_2S_3} \\ &= e^{K(S_1+S_3)} + e^{-K(S_1+S_3)} \end{aligned}$$

If $S_1 = S_3 = +1$

$$e^{K'} * e^{2g(K)} = e^{2K} + e^{-2K} \tag{3}$$

If $S_1 = -S_3 = +1$

$$e^{-K'} * e^{2g(K)} = 2 \tag{4}$$

Using Eq.(4) we obtain g(K)

$$g(K) = \frac{1}{2}K^{'} + \frac{1}{2}\ln 2$$

Next using Eq. (3) we obtain K'

$$K^{'} = \frac{1}{2}\ln\{\cosh(2\mathrm{K})\}$$

Thus K' is related to the original coupling K by a non-linear transformation. We denote this as $K' = f(K)$. Near the fixed point $K = K^* + \epsilon$,

$$\begin{aligned} K' &= f(K^* + \epsilon) \\ K^* + \epsilon' &= f(K^*) + \epsilon f'(K^*) \end{aligned}$$

As $K^* = f(K^*)$ near a fixed point K* we have

$$\epsilon' = \epsilon f'(K^*)$$

which is a linear transformation that resembles

$$\epsilon' = \lambda\epsilon$$

where $\lambda = \tanh(2K^*)$.

There are two solutions for the equation $K^* = \ln\{\cosh(2K^*)\}/2$, which are known as fixed points, $K^* = 0$ and $K^* = \infty$ with $\lambda = 0$ and $\lambda = 1$ respectively. For phase transition λ must be greater than unity. This proves the well established result that there is no phase transition for 1D Ising spin model. There is another way to see this. After applying the renormalization transformation n times, the mapping $K^{n-1} \rightarrow K^{(n)}$ can be obtained from the recurrence relation,

$$K^{(n)} = \frac{1}{2}\ \ln\ \{\cosh(2\mathrm{K}^{(n-1)})\} \tag{5}$$

where $K^{(0)} = K$.

$$\text{Let } \zeta = \tanh(\mathrm{K}) \tag{6}$$

$$\text{therefore } K' = \frac{1}{2}\ln(\frac{1+\zeta^2}{1-\zeta^2}) \tag{7}$$

$$\begin{aligned} \text{Hence, } \zeta' = \tanh(\mathrm{K'}) &= \tanh\{\frac{1}{2}\ln(\frac{1+\zeta^2}{1-\zeta^2})\} \\ &= \frac{\exp\Big(\frac{1}{2}\ln\Big(\frac{1+\zeta^2}{1-\zeta^2}\Big)\Big) - \exp\Big(-\Big(\frac{1}{2}\ln\Big(\frac{1+\zeta^2}{1-\zeta^2}\Big)\Big)\Big)}{\exp\Big(\frac{1}{2}\ln\Big(\frac{1+\zeta^2}{1-\zeta^2}\Big)\Big) + \exp\Big(-\Big(\frac{1}{2}\ln(\frac{1+\zeta^2}{1-\zeta^2}\Big)\Big)\Big)} \\ &= \frac{\sqrt{\frac{1+\zeta^2}{1-\zeta^2}} - \sqrt{\frac{1-\zeta^2}{1+\zeta^2}}}{\sqrt{\frac{1+\zeta^2}{1-\zeta^2}} + \sqrt{\frac{1-\zeta^2}{1+\zeta^2}}} \\ &= \frac{(1+\zeta^2) - (1-\zeta^2)}{(1+\zeta^2) + (1-\zeta^2)} \end{aligned}$$

Thus,

$$\begin{aligned} \zeta' &= \zeta^2 \\ \tanh(\text{K'}) &= \tanh(\text{K})^2 \end{aligned} \tag{8}$$

Since $\tanh(\text{K}) < 1$, $\tanh(\text{K}^n)$ tends to zero as $n \longrightarrow \infty$. This suggests that the effective coupling gets weaker with each decimation and we are left with a non-itneracting system which will show no phase transition.

Free Energy and Scaling equation

The equation for free energy per spin is given by,

$$f_N(K) = \frac{1}{N} ln \sum_{[s]} exp[-H_N(K)]^1$$

If we use the expression for Hamiltonian in K′ post decimation from Eq.(2) we can write,

$$\begin{aligned} f_N(K) &= \frac{1}{N}\ln\sum_{[s]}\exp[-H_{\frac{N}{2}}(K') + Ng(K)] \\ f_N(K) &= f_{\frac{N}{2}(K')} + \frac{1}{N}\ln\sum_{[s]}\exp[Ng(K)] \\ f_N(K) &= f_{\frac{N}{2}(K')} + g(K) \end{aligned}$$

$$f_{N/2}K^{'} = 2\{f_N(K) - g(K)\} \tag{9}$$

In the thermodynamic limit, the functional relation in Eq. (9) leads to the scaling equation (for $f(K) = lim_{N\to\infty} f_N(K)$),

$$f(K') = 2\{f(K) - g(K)\} \tag{10}$$

We can prove that the scaling equation obtained is unique as follows,

$$\begin{aligned} f_-(K) &= f_1(K) - f_2(K) \\ f_-(K') &= f_1(K') - f_2(K') \\ &= 2\{f_1(K) - g(K)\} - 2\{f_2(K) - g(K)\} \\ f_-(K) &= \frac{1}{2} f_-(K') \\ f_-(K) &= \frac{1}{2^n} f_-(K^{(n)}) \end{aligned}$$

$$\begin{aligned} \lim_{n\to\infty} f_-(K^{(n)}) &= 0 \quad \text{and} \quad f_-(0) = 0 \longrightarrow 0 \\ \therefore f_-(K) &= 0 \\ \text{Hence,} \quad f_1(K) &= f_2(K) \end{aligned}$$

Free energy solution

Henceforth in this section, we intend to find a solution to the free energy equation obtained in the previous sections imposing boundary conditions as follows,

$$\begin{aligned} f(0) &= \ln 2 \\ \text{Thus,} \quad f_-(K) &= \frac{1}{2} f(K') + g(K) \end{aligned}$$

[1] free energy, $f = +\frac{1}{N}\ln$ Z, as taken in this document

Next decimation step gives,

$$\begin{aligned}
f_-(K') &= \frac{1}{2}f(K'') + g(K') \\
2(f_-(K) - g_-(K)) &= \frac{1}{2}f(K'') + g(K') \\
f_-(K) - g_-(K) - \frac{1}{2}g(K) &= \frac{1}{2}f(K'') \\
f_-(K) &= \frac{1}{2}f(K'') + g_-(K) + \frac{1}{2}g(K) \\
f_-(K) &= \frac{1}{2}f(K'') + \sum_{i=0}^{1}\frac{g(K)}{2^i} \\
\text{Generalising,} \quad f_-(K) &= \frac{1}{2^n}f(K(n)) + \sum_{i=0}^{n-1}\frac{g(K)}{2^i}
\end{aligned}$$

let $n \longrightarrow \infty$ and $K^{(n)} \longrightarrow 0$ and defining

$$\begin{aligned}
h(K) &= \lim_{n\longrightarrow\infty}\frac{f(k^{(n)})}{2^n} \\
\text{we get, } f(K) &= h(K) + \sum_{i=0}^{1}\frac{g(K)}{2^i}
\end{aligned}$$

If f(0)= finite then h(K)=0

$$\begin{aligned}
\therefore f(K) &= \sum_{i=0}^{\infty}\frac{g(K(^m))}{2^m} \\
&= g(K) + \frac{g(K')}{2} + \frac{g(K'')}{2^2} + \frac{g(K''')}{2^3} + ... \\
&= \frac{1}{2}\ln 2 + \frac{1}{2}K'\frac{1}{2^2}\ln 2 + \frac{1}{2^2}K'' + \frac{1}{2^3}\ln 2 + \frac{1}{2^3}K''' + ... \\
&= \left(\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + ...\right)\ln 2 + \sum_{n=1}^{\infty}\frac{K^n}{2^n} \\
&= \frac{\frac{1}{2}}{1-\frac{1}{2}}\ln 2 + \sum_{n=1}^{\infty}\frac{K^n}{2^n}{}^{2}
\end{aligned}$$

$$= \ln 2 + \sum_{n=1}^{\infty}\frac{K^n}{2^n}$$

Using the recurrence relation for K^n from Eq. (5) we have,

$$f(K) = \ln 2 + \sum_{n=0}^{\infty}\frac{\frac{1}{2}\ln\cosh(2K^{n-1})}{2^n}$$

Eqs. (6), (7) and (8) give us the liberty to write,

$$\begin{aligned}
f(K) &= \ln 2 + \sum_{n=1}^{\infty}\frac{\ln\left(\frac{1+\zeta^{2^n}}{1-\zeta^{2^n}}\right)}{2^{n+1}} \\
f(K) &= \ln 2 + \ln\left(\prod_{n=1}^{\infty}\left(\frac{1+\zeta^{2^n}}{1-\zeta^{2^n}}\right)^{1/2^{n+1}}\right) \qquad -1 < \zeta < 1 \qquad (11)
\end{aligned}$$

[2](sum of an infinite geometric progression $= \frac{a}{1-r}$, $a = \frac{1}{2}$ being the first term and $r = \frac{1}{2}$ being common ratio)

Applying the trigonometric identity,

$$\frac{1}{\sqrt{1-x^2}} = \prod_{n=0}^{\infty}\left(\frac{1+x^{2^n}}{1-x^{2^n}}\right)^{1/2^{n+1}} \qquad -1<x<1$$

we have the following equations,

$$\begin{aligned}
f(K) &= \ln 2 + \ln\left(\frac{1}{\sqrt{1-\zeta^2}}\right) \\
&= ln2 - \ln(\sqrt{1-\zeta^2}) \\
&= \ln\left(\frac{2}{\sqrt{1-\zeta^2}}\right) \\
f(K) &= \ln\left(\frac{2}{\sqrt{1-\tanh^2(K)}}\right)
\end{aligned}$$

$$\text{Hence, } f(K) = \ln(2\cosh(\mathrm{K})) \tag{12}$$

Note that this is the exact relation for free energy of 1-D Ising Model (if $f = +\frac{1}{N} lnZ$). Further we show that it satisfies the scaling Eq.(10).

$$\begin{aligned}
\text{R.H.S.} &= 2\left\{\ln(2\cosh(\mathrm{K})) - \frac{1}{2}\ln 2 - \frac{1}{2}K'\right\} \\
&= -\ln 2 - K' + 2\ln(2\cosh(\mathrm{K})) \\
&= -\ln 2 - K' + 2\ln 2 + 2\ln(\cosh(\mathrm{K})) \\
&= \ln 2 - K' + 2\ln(\cosh(\mathrm{K})) \\
&= \ln 2 - \frac{1}{2}\ln\ (\cosh(2\mathrm{K})) + 2\ln(\cosh(\mathrm{K}))
\end{aligned}$$

$$\begin{aligned}
\text{L.H.S.} &= \ln(2\cosh(\mathrm{K'})) \\
&= \ln 2 + \ln(\cosh(\mathrm{K'})) \\
&= \ln 2 + \ln(\frac{e^{K'} + e^{-K'}}{2}) \\
&= \ln 2 - \ln 2 + \ln(\exp(\frac{1}{2}\ln(\cosh(2\mathrm{K}))) + \exp(\exp(-\frac{1}{2}\ln(\cosh(2\mathrm{K})))) \\
&= \ln(\cosh(2\mathrm{K})^{\frac{1}{2}} + \frac{1}{\cosh(2\mathrm{K})^{\frac{1}{2}}}) \\
&= \ln(\cosh(2\mathrm{K}) + 1) - \frac{1}{2}\ln\cosh(2\mathrm{K}) \\
&= \ln(2\cosh^2(2K)) - \frac{1}{2}\ln(\cosh(2\mathrm{K})) \\
&= \ln 2 + 2\ \ln(\cosh(2\mathrm{K})) - \frac{1}{2}\ln(\cosh(2\mathrm{K}))
\end{aligned}$$

As we can see from the above equations, L.H.S.= R.H.S., this proves that the free energy solution determined by Eq.(12) satisfies the scaling equation. $f(K) = 2\quad \ln(\sinh(\mathrm{K}))$ also satisfies the scaling equation but it does not satisfy the boundary condition f(0)=ln2. Thus, it is not a solution.

We next discuss the more complex (unequal J_i) one-dimensional Ising models.

Alternate Ising model

Here the $K_i's$ are arrranged in the manner shown in Fig.(3).

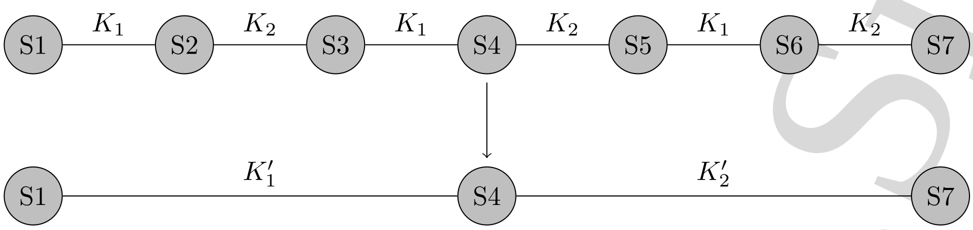

Figure 3: Similar Decimation for Alternate Ising Model

In order to adopt a similar decimation procedure we need to consider four spins at a time. This is illustrated in Fig.(3). Using this procedure,

$$\begin{aligned} e^{K_1' S_1 S_4} * e^{g_1} &= \sum_{S_2, S_3} e^{K_1 S_1 S_2} * e^{K_2 S_2 S_3} * e^{K_1 S_3 S_4} \\ &= \sum_{S_3} e^{K_1 S_3 S_4} * 2\cosh(K_1 S_1 + K_2 S_3) \\ &= e^{K_1 S_4} * 2cosh(K_1 S_1 + K_2) + e^{-K_1 S_4} * 2\cosh(K_1 S_1 - K_2) \end{aligned}$$

Like in the previous section, we consider $S_1 = S_4 = +1$ to obtain ,

$$e^{K_1'} * e^{g_1} = e^{K_1} * 2\cosh(K_1 + K_2) + e^{-K_1} * 2\cosh(K_1 - K_2) \quad (13)$$

and $S_1 = -S_4 = +1$ to obtain,

$$e^{-K_1'} * e^{g_1} = e^{-K_1} * 2\cosh(K_1 + K_2) + e^{K_1} * 2\cosh(K_1 - K_2) \quad (14)$$

Using Eqs.(13) and (14), and cosh(x)=cosh(-x) we obtain,

$$e^{2K_1'} = \frac{e^{K_1}\cosh(K_1 + K_2) + e^{-K_1}\cosh(K_1 - K_2)}{e^{K_1}\cosh(K_1 - K_2) + e^{-K_1}\cosh(K_1 + K_2)}$$

Using Componendo and Dividendo, we write

$$tanh(K_1') = \frac{e^{K_1}\{cosh(K_1 + K_2) - cosh(K_1 - K_2)\} + e^{-K_1}\{cosh(K_1 - K_2) - cosh(K_1 + K_2)\}}{e^{K_1}\{cosh(K_1 + K_2) + cosh(K_1 - K_2)\} + e^{-K_1}\{cosh(K_1 - K_2) + cosh(K_1 + K_2)\}}$$

and employing the addition properties of cosh(x),

$$\begin{aligned} cosh(K_1 + K_2) &= cosh(K_1)cosh(K_2) + sinh(K_1)sinh(K_2) \\ cosh(K_1 - K_2) &= cosh(K_1)cosh(K_2) - sinh(K_1)sinh(K_2) \end{aligned}$$

We have,

$$\tanh(K_1') = \tanh^2(K_1)\tanh(K_2)$$

One similarly obtains,

$$\tanh(K_2') = \tanh^2(K_2)\tanh(K_1)$$

The fixed points are,$\{K_1^*, K_2^*\} = \{0, 0\}$ or $\{\infty, \infty\}$.

Note that we need to block spins in a judicious way. If we block them in a non similar fashion, i.e. if the new lattice is not alternate (see Fig.(4)), then,

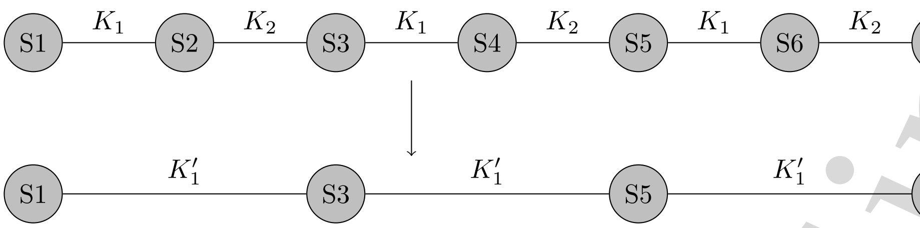

Figure 4: Non similar decimation

$$\tanh(K_1') = \tanh(K_1)\tanh(K_2)$$

In this case a fixed point discussion is not possible as K_2' does not exist. However, the free energy is the same in either case.

Fibonacci chain Ising model

In this section we consider a fibonacci series where two $b's$ are never adjacent *i.e.* the nearest neighbour of 'b' is always 'a' (see Fig.(5)). We suggest a method to generate the decimation procedure below.

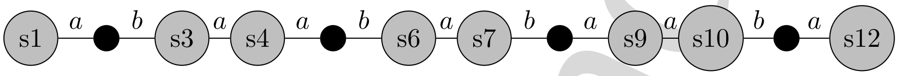

Figure 5: Fibonacci Ising model

Matrix method for generation

It is well known that the Fiboanacci chain can be generated by setting up a number of rules for rabbit procreation, better known as Fibonacci Rabbits[4]. In the present case we generate it by the mathematical operation shown below. Note that $N^{(0)}$ denotes a vector AB. $N^{(i)}$ denotes the resulting vector after $N^{(0)}$ has been operated 'i' times by M, a matrix operator to generate the Fibonacci Ising chain.

$$M = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \quad N^{(0)} = \begin{bmatrix} N^{(0)_A} \\ N^{(0)_B} \end{bmatrix} = \begin{bmatrix} A \\ B \end{bmatrix} : AB$$

$$MN^{(0)} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} A+B \\ A \end{bmatrix} : ABA = N^{(1)}$$

$$MN^{(1)} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} A+B \\ A \end{bmatrix} = \begin{bmatrix} A+B+A \\ A+B \end{bmatrix} = \begin{bmatrix} N^{(2)_A} \\ N^{(2)_B} \end{bmatrix} = N^{(2)} : ABAAB$$

$$MN^{(2)} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} A+B+A \\ A+B \end{bmatrix} = \begin{bmatrix} A+B+A+A+B \\ A+B+A \end{bmatrix} = \begin{bmatrix} N^{(3)_A} \\ N^{(3)_B} \end{bmatrix} = N^{(3)} : ABAABABA$$

and so on.

For general iteration,

$$\begin{aligned} N_A^{(n+1)} &= M_{11}N_A^{(n)} + M_{21}N_B^{(n)} \quad M_{11} = M_{21} = M_{12} = 1 \\ N_B^{(n+1)} &= M_{12}N_A^{(n)} + M_{22}N_B^{(n)} \quad M_{22} = 0 \end{aligned}$$

as $n \longrightarrow \infty$, the ratio $r = \lim_{n\longrightarrow\infty} N^{(n)_A}/N^{(n)_B} \longrightarrow (\sqrt{5}+1)/2$. Here N_A, N_B are length scales of bond A and B respectively.

Decimation method

Consider now the Ising Chain shown in Fig.(6):

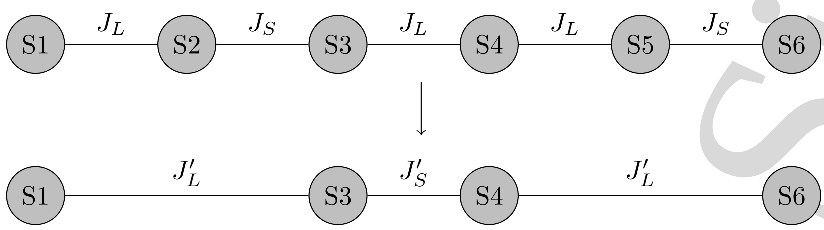

Figure 6: Self-similar blocking for Fibonacci chain

Let $K_i = J_i/kT$

$$\begin{aligned} H_N(K) &= -\sum_{i=1}^{N-1} K_i S_i S_{i+1} \\ e^{K'_L S_1 S_3} * e^{g_1} &= \sum_{S_2=+1}^{-1} e^{K_L S_1 S_2} * e^{K_S S_2 S_3} \end{aligned}$$

As in the previous sections, let $S_1 = S_3 = +1$ we obtain,

$$e^{K'_L} * e^{2g_1} = e^{K_L+K_S} + e^{-(K_L+K_S)} \tag{15}$$

and $S_1 = -S_3 = +1$ yields,

$$e^{-K'_L} * e^{2g_1} = e^{K_L-K_S} + e^{-(K_L-K_S)} \tag{16}$$

Using Eqs.(15) and (16), and $cosh(x) = cosh(-x)$ leads to,

$$e^{2K'_L} = \frac{cosh(K_L+K_S)}{cosh(K_L-K_S)}$$

Using Componendo and Dividendo, we get

$$\begin{aligned} tanh(K'_L) &= \frac{cosh(K_L+K_S) - cosh(K_L-K_S)}{cosh(K_L+K_S) + cosh(K_L-K_S)} \\ &= tanh(K_L)tanh(K_S) \\ \text{and } g_1 &= \frac{1}{2}K'_L + \frac{1}{2}ln(2cosh(K_L-K_S)) \end{aligned}$$

For the ordered case ($K_L = K_S$) the tansformation reduces to $tanh(K') = tanh^2(K)$ and g_1 is given by the same expression as Eq.(18). Further, $K'_S = K_L$ and $g_2 = 0$. Hence, all the above equations are consistent with ordered case. The fixed points in this case are,$\{K_1^*, K_2^*\} = \{0, 0\}$ or $\{\infty, \infty\}$.

We designate bond lengths, distance between two neighbouring sites in the lattice before and after first decimation process, for the two types of bonds by two sets of variables. Here, the new lengths are L′ for K'_1 and S′ for K'_2 corresonding to the old lengths, L for K_1 and S for K_2. Self similarity is preserved if the new lengths follow the following relation.

$$\frac{L'}{S'} = \frac{L}{S}$$

But $L' = L + S$ and $S' = L$. Thus,

$$\begin{aligned} 1 + \frac{S}{L} &= \frac{L}{S} = x \\ 1 + \frac{1}{x} &= x \end{aligned}$$

which leads to $x = (\sqrt{5}+1)/2$, the golden ratio.

Conclusion

This decimation approach is perhaps the simplest version of RNG. Its extension to higher dimension however gets tricky. The solution to this problem uses the Migdal Kadanoff transformation[5],[6]. Interestingly, this decimation procedure inspired similar work in quantum systems. We hope to describe this quantum version introduced by Bhat, Singh and Subbarao[7] in the future.

Appendix

Partion Function and Free Energy for 1D Ising model

The partition function for this system can be derived as $follows$:

$$Z_N = \sum_{s_1,s_2,s_3\ldots} exp(-H) \qquad (kT = 1) \tag{17}$$

$$Z_N = \sum_{s_1,s_2,s_3\ldots s_n} exp(K) \sum_{i=1}^{n} (S_i S_{i+1}) \qquad (kT = 1) \tag{18}$$

$$Z_N = \sum_{s_1,s_2,s_3\ldots s_{n-1}} exp(K) \sum_{i=1}^{n-1} (S_i S_{i+1}) \sum_{s_n,s_n+1=1}^{s_n,s_n+1=-1} exp(K S_n S_{n+1}) \tag{19}$$

$$Z_N = Z_{N-1} \sum_{s_n,s_n+1=1}^{s_n,s_n+1=-1} exp(K S_n S_{n+1}) \tag{20}$$

$$Z_N = Z_{N-1} \sum_{s_n=1}^{s_n=-1} (2\mathrm{cosh}\mathrm{K} S_n) \tag{21}$$

$$Z_N = \sum_{S_i=-1}^{S_i=1} (2^N \cosh(\mathrm{K} S_i)^N) \tag{22}$$

As $S_i = \pm 1$, and cosh function is independent of sign of the argument, hence the above equation can be written as

$$Z_N = (2^{N+1} \cosh(\mathrm{K})^N) \tag{23}$$

Free Energy equations would follow as,

$$\frac{F_N}{kT} = -\mathrm{kTlogZ}$$

$$\frac{F_N}{kT} = -(N+1)ln2 - Nln\{cosh(K)\} \tag{24}$$

$$f_N = \frac{F_N}{NkT} = -ln2 - ln\{cosh(K)\} \qquad (\text{Taking } \frac{1}{N} \to 0) \tag{25}$$

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