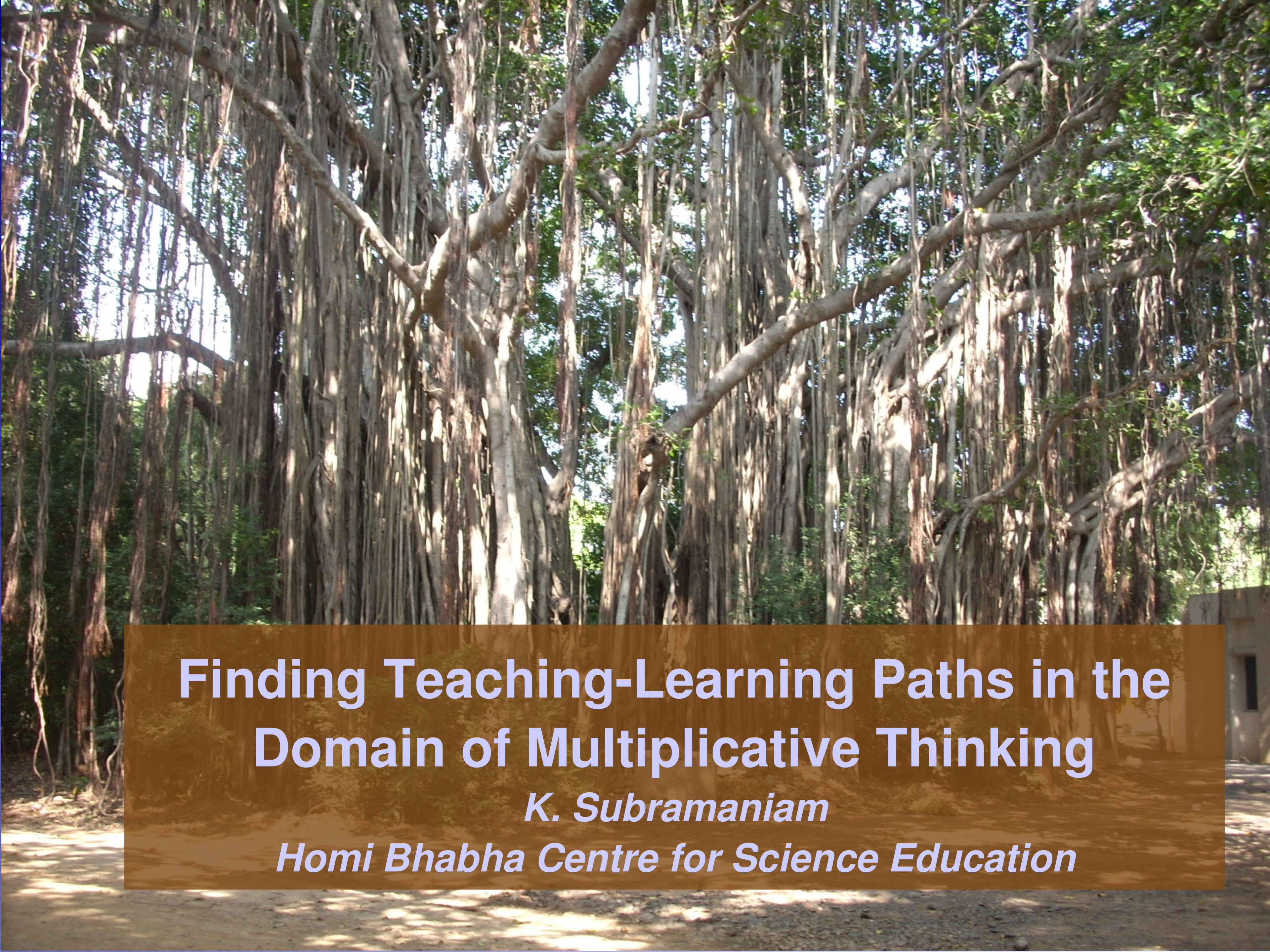

Finding Teaching-Learning Paths in the Domain of Multiplicative Thinking

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Overview of my talk

On Teaching
-Learning Paths

- What is multiplicative thinking?
- Summary of the research on the semantics of rational numbers (80s and early 90s)
- Decline of research activity
- What is needed to develop learning paths?
- Issues of culture
- From schemes to symbolic mathematics
  - Visual representations
  - Algebraic aspects of the fraction notation

What is Multiplicative Thinking?

On Teaching
-Learning Paths

- It involves attenting to the multiplicative relation between quantities or magnitudes
- Many contexts require one to do this:
  - Situations involving measurement
  - Situations involving proportionality
  - Situations where two or more quantities co-vary
- Multiplicative Thinking is cognitively demanding because it involves grasping relationship between quantities.

Multiplicative Thinking as a mental construct

- A hypothesized mental structure, explains certain spontaneous strategies that children adopt
- Is different from and more complex than additive thinking, which is acquired earlier.
  - Example: If 12 ladoos cost Rs 32, what is the cost of 15 ladoos?

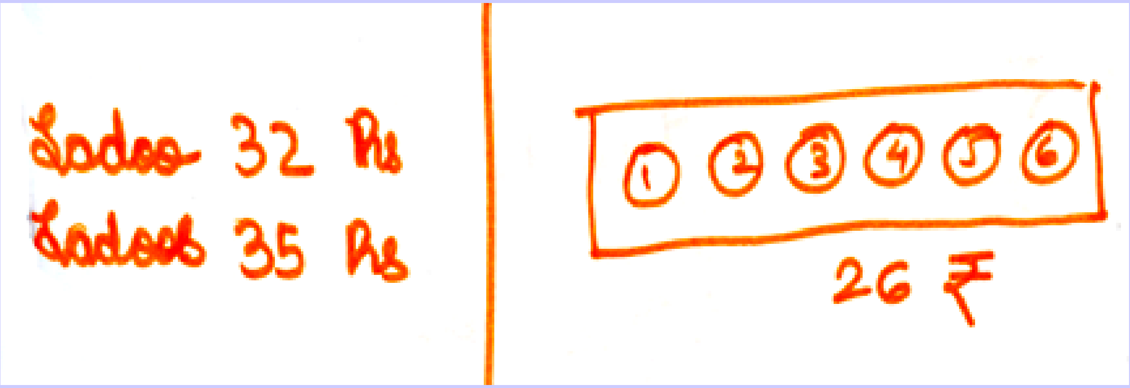
Cost of 15 ladoos: student responses

- Student A :

If 12 ladoos cost Rs 32, what is the cost of 15 ladoos?

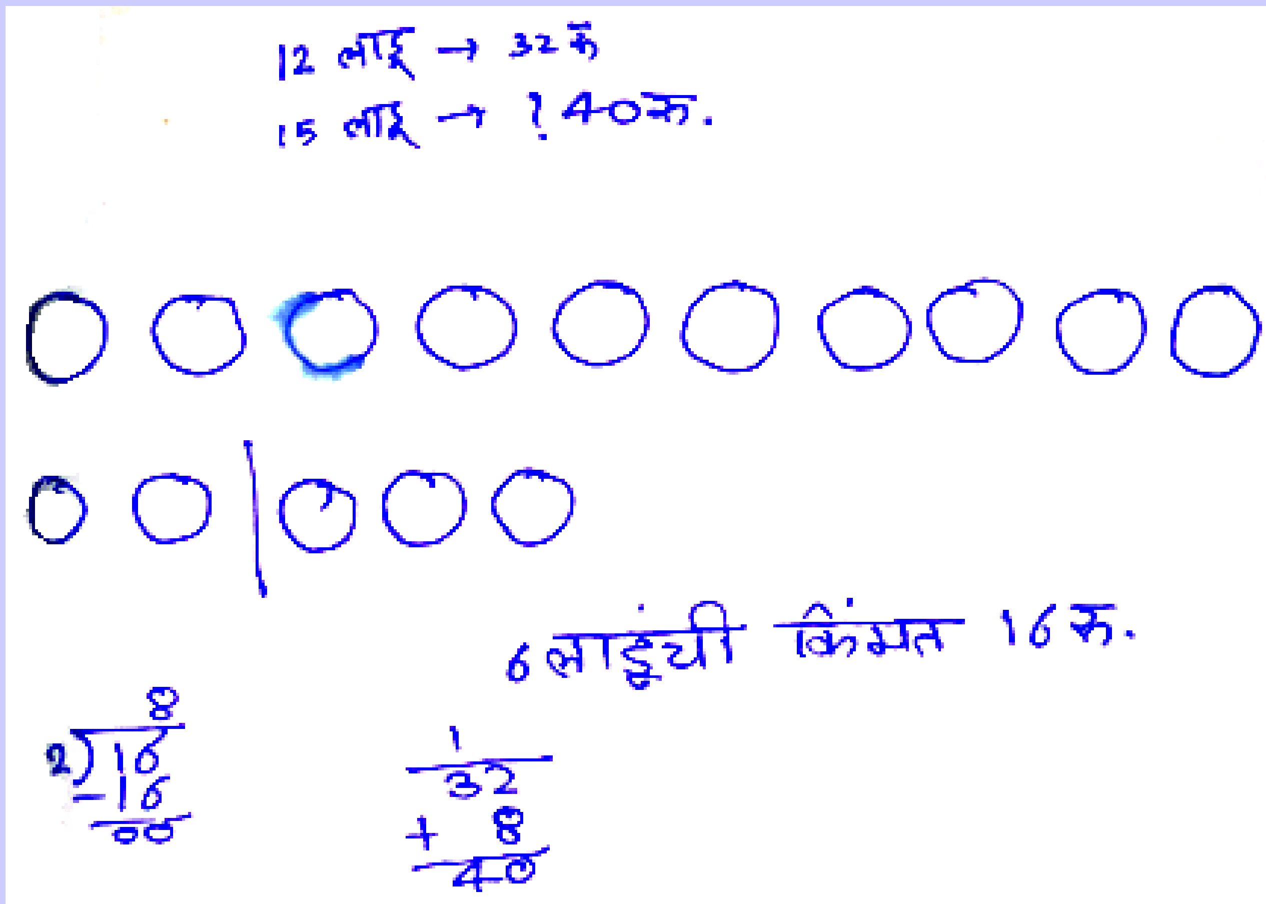
15 ladoos cost Rs 35

6 ladoos cost Rs 26



Cost of 15 ladoos: student responses

- Student B : Struggles to find the *cost of one ladoo*.
  Asked if he can solve the problem without finding the cost of one ladoo.



A second meaning: Multiplicative Thinking as a connecting mathematical theme

- Forms the backbone of the mathematics curriculum. Includes important and inter-connected ideas such as
  - Multiplication and Division
  - Fractions, decimals
  - Ratio, percentages
  - Proportionality
  - Linear functions

Why is Multiplicative Thinking Important?

- Successful application of multiplicative thinking to situations requires
  - Reasoning about quantities and their relationships
  - Ability to represent the relationships mathematically and facility with generating and transforming representations
- The fraction notation is the main tool in representing multiplicative relationships and manipulating/ calculating with them.

Fractions and Rational Numbers

- Fractions can be thought of as
  - non-negative rational numbers
    (a subset of the rational numbers)
  - an algebraic notation for the result of the division operation.
    (may also include irrational numbers such as π/4)

An exclusive part-whole interpretation is not pedagogically fruitful.

Place of Fractions in The Curriculum

On Teaching
-Learning Paths

- The traditional justification
  - Fractions form the conceptual basis for decimals and percentages
  - The arithmetic of fractions (rational numbers) is needed for algebra
- These reasons are still valid !

Place of Fractions in The Curriculum

- Other important reasons:
  - Rational numbers greatly expand the scope of what can be quantified; they also allow quantification of relationships: larger, sweeter, denser, more crowded, etc.
  - Fractions provide a useful didactic context for developing multiplicative reasoning.

  Freudenthal (1983): “Fractions are the phenomenological source of the rational number”

Research on the semantic analysis of Rational Numbers: late 70s to the 90s

Research on rational number learning: late 70s to the 90s

- Most distinguishing feature was the semantic analysis of rational numbers
- Researchers studying fraction learning often also studied proportional thinking
- Largely from North America and Western Europe

Some important references

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- Confrey, J. et al. Rational Number Research Database. http://gismo.fi.ncsu.edu

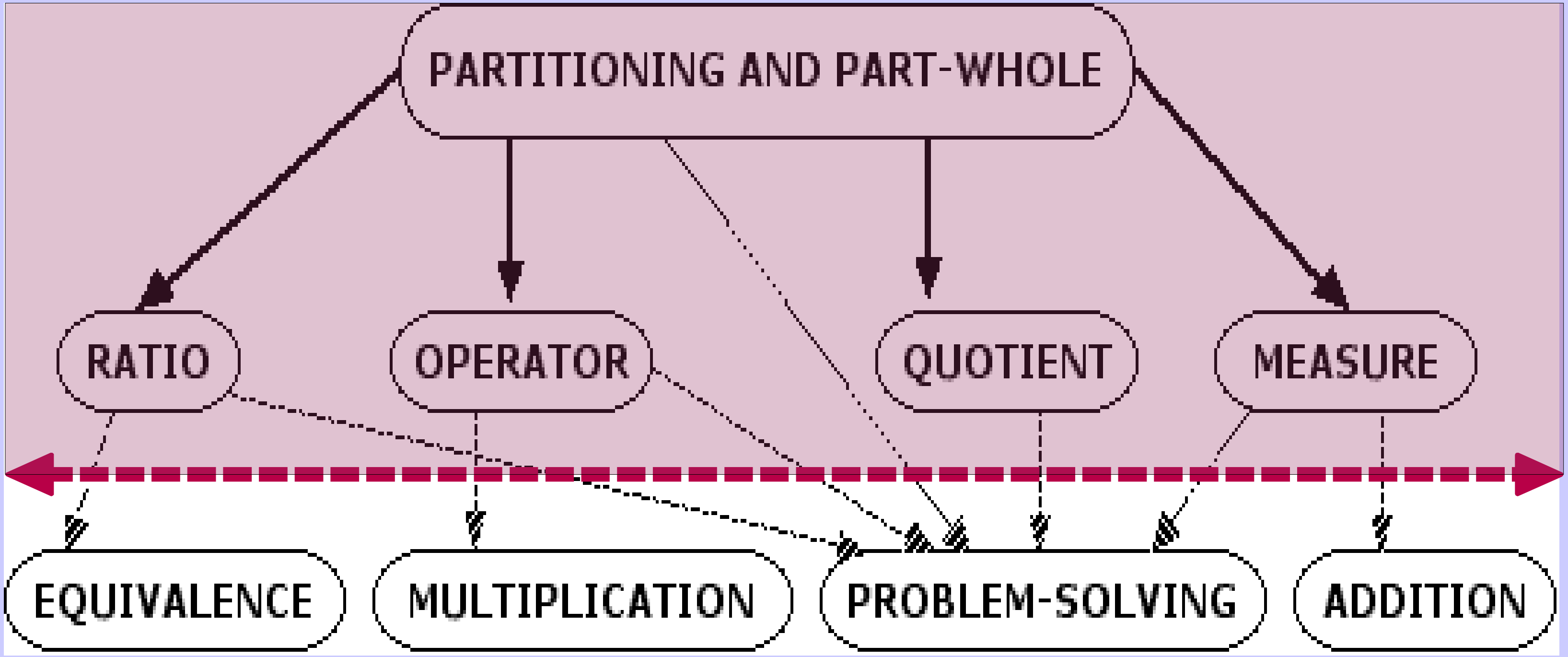
Semantic analyses of rational numbers

- Four main aspects that we will consider:
  - Subconstruct theory (Thomas Kieren, 1976, 1980)
  - The multiplicative conceptual field (Ohlsson 1988, Schwarz 1988, Vergnaud 1988)
  - Semantics of partitioning and unitizing (Behr et al., 1992, The Rational Number Project)
  - Exploring the equal sharing context as a starting point for teaching fractions. (Streefland, 1993)

Rational number subconstructs

- Research before the 70's:
  - Hierarchy of skills needed for the arithmetic of fractions
  - How to teach them
  - How fraction manipulatives can help
- Kieren's writings suggested that what was difficult about fractions was
  - The meaning of fractions
  - The fact that fractions seemed to have different meanings, all of which students needed to learn.

Rational number subconstructs



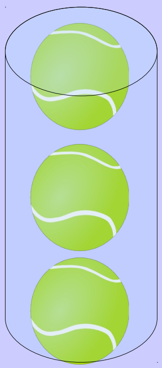
The multiplicative conceptual field

Semantic analysis was extended to the multiplication operation

- Schwartz (1988): Referent changing nature of the multiplication operation; Intensive quantity as quantifying a relation
- Vergnaud (1988): Analysis of multiplicative situations; Scalar and Functional relations
- Post et al. (1993): Need to go beyond multiplication as repeated addition

Semantic analysis of the unit concept

- Simple unit: ***3 [(1-unit)s]***

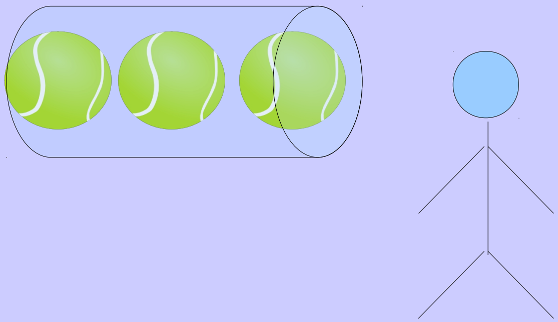


- Composite unit or *unit of units*:

  1 [(3-unit)s]

- Ratio unit (unit of composite units) or *unit of units of units:*

$$\mathbf{1}\ \left[\ \frac{\textbf{(3-unit)s}}{\textbf{(1-unit)}}\ \right]$$



Semantic analysis of the unit concept

- Flexible unitizing is important:

  1½ dozen bananas

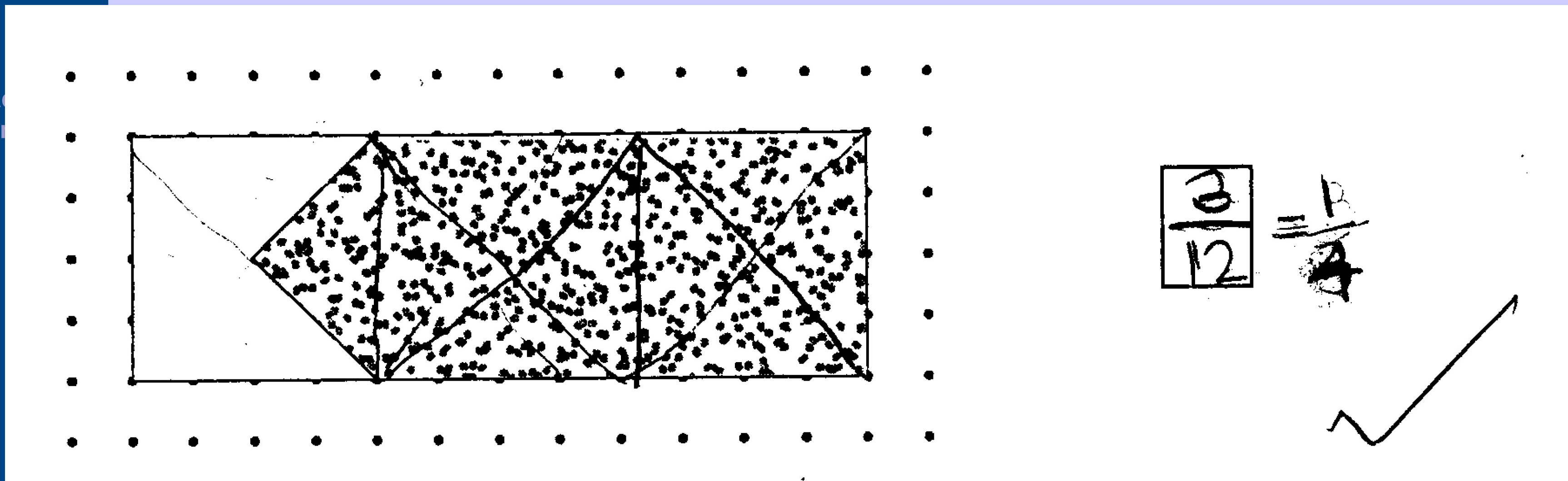
  = 18 bananas

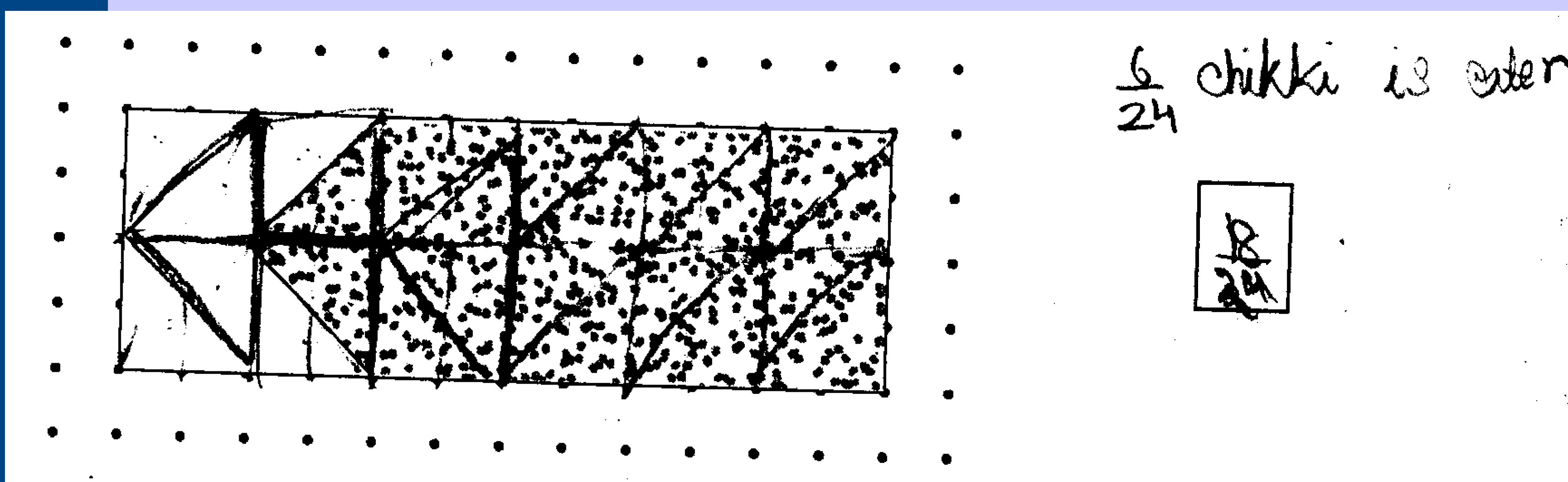
  = 3 (½ dozens)

  = ...

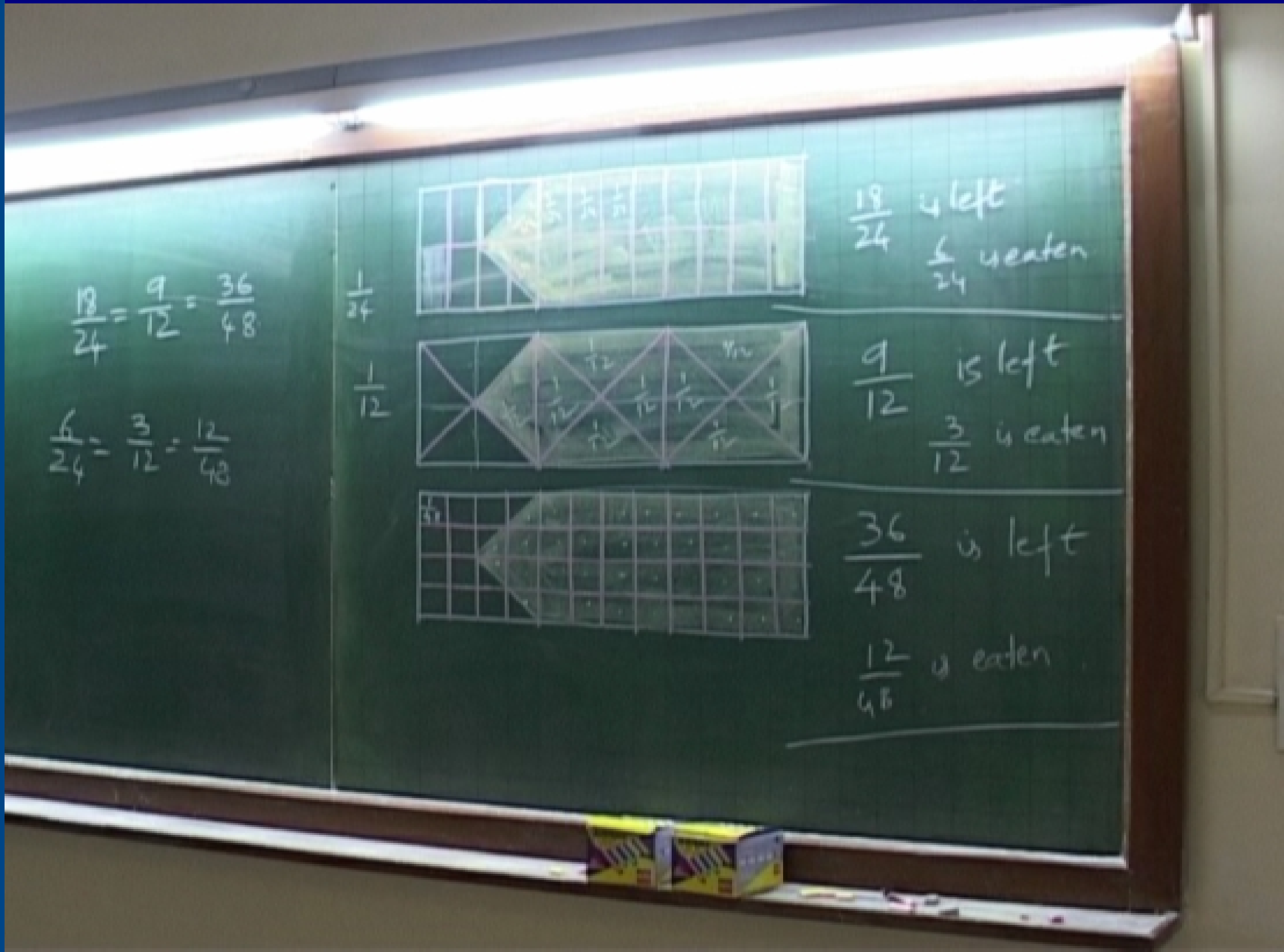
Flexible unitizing with partitioning

On Tea
-Learni

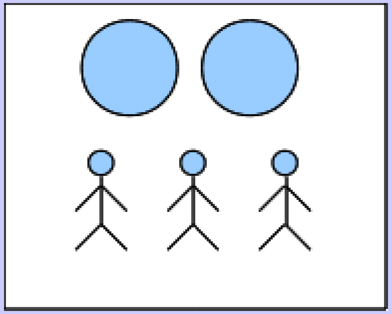


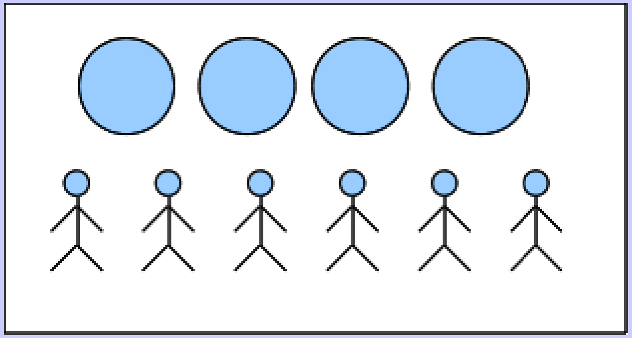


Flexible unitizing with partitioning



Equal sharing as a context for teaching fractions





- Freudenthal (1973): fraction as quotient is the most meaningful setting for studying rational numbers
- Explored deeply through teaching studies by Streefland.
- Explored extensively including at Eklavya and HBCSE.

Limitations of the research on rational numbers

- Although period saw intenstive research, its impact on curricula and actual teaching was limited and continues to be so.
- Behr et al. (1992) remarked that teaching intervention studies were limited. (Streefland's work was an exception; Better after 90s)
- Lamon (2007) “ It was difficult (at that time) to see where the field was going”
- Davis, Hunting & Pearne (1993): “No real progress was being made”

Decreased attention to the domain after 90s

- The number of publications decreased
  - PME 20 in 1996: 15 articles on rational number and proportion out of 240.
  - PME 30 in 2006: 10 articles on number concepts and operations out of 478.
- Lamon (2007): The present “crisis” in the field.
  - Apart from the complexity of the domain, there is a need for the researcher's long term commitment.

Themes of research on rational numbers after 2000

- Extensions and filling gaps of previous research
- Teaching intervention studies
- Children's schemes related to rational numbers
- Teachers' knowledge (impact of Liping Ma's study and other Asian studies)
- Conceptual differences between whole numbers and rational numbers

What is needed for progress

- Lamon (2007): recommends longitudinal teaching intervention studies
- Nunes and Bryant (2009) : recommend teaching experiment studies with suitable controls
- Learning progressions/ trajectories important for impact on curricula

Comparison with whole numbers, addition and subtraction

- In contrast to the domain of rational numbers, whole number learning, addition and subtraction has witnessed a consolidation.
- General consensus about broad learning progression/ teaching learning trajectory
- Example: Sarama and Clements (2009): *Early Childhood Mathematics Education Research: Learning Trajectories for Young Children*.

Learning trajectories/ progressions

- Learning progressions in science education (JRST, special issue in 2009)
- Learning trajectories in mathematics education
  - Simon 1995: 'hypothetical learning trajectory' a construct useful for a teacher, to integrate learning goals with the students' own mathematical thinking.
  - Confrey et al. 2009: 'Learning trajectory' a construct that can help integrate research findings about teaching and learning in a topic domain

Learning trajectories for rational numbers?

- Freudenthal Institute (van Galen et al., 2005): A teaching-learning trajectory for the topics of fractions, percentages, decimals and proportions (English transln: 2008)
- Confrey et al. (2009): Learning trajectory as a tool to organize the Rational Number Research database
- Lamon (1999, 2nd edn. 2006): *Teaching fractions and ratios for understanding*: Gathers together many results of the earlier phase of research in the form of learning activities for students and teachers, and examples of student work.

Three inputs to identifying learning progressions

Knowledge about

- Children's intuitive capacities (action schemes)
- Support for learning in the culture
- How children bridge action schemes and symbolic routines

The Concept of Scheme

Scheme: A hypothesized cognitive structure indicating organization of potential actions on concrete, symbolic or mental objects trans-forming them in meaningful, goal-directed ways.

- Schemes
  - Have an assimilation structure
  - Have an action structure
  - Include sensitivity to result of the action
- Schemes undergo elaboration and integration, become more complex and powerful

The explanatory role of “scheme”

- Identify important transitions that children make
- Hypothesize the emergence of an underlying cognitive structure that accounts for this transition
- Example: Transition in addition strategy from “count all” to “count on” with and without concrete support.

Children's multiplicative schemes

- Equipartitioning scheme: extensive studies: Pothier & Sawada 1983, Empson et al. 2005, Confrey et al. 2009)
- Scheme of correspondence: Nunes and others: several studies,
  - building up strategies have been studied over many years
- Steffe and Olive 2010: “Children's fraction schemes arise as a result of modifications of their counting schemes.”

Language issues

- Fractions are read in particular ways in different Indian languages:
  - “three by five”; “three upon five” (English: quotient)
  - “three shared among/ divided by five” (Hindi, Urdu: quotient)
  - “three out of five” (Tamil: part-whole)
  - “three upon five” (Marathi: ?)

(PME 34, 2010, Research Forum on Mathematics and Culture)

Support from the culture

- Large body of research dealing with out of school knowledge about proportions
- Participation of the household in the economy is a source of mathematical knowledge of various kinds.
- Children too participate.
- The forms of participation vary.

"But cultural knowledge is not merely a vehicle to deliver formal mathematics."

Culture and mathematics learning

"We cannot simply mine what is present in the culture as a resource to push a particular curricular agenda.... (We need) to reflect more deeply about the relation between academic and cultural knowledge. In the long run, if a form of knowledge is to survive and flourish, it must have deep roots in the culture."

(*Learning Curve*, 2010, Special issue on mathematics Education)

Support from the culture

- The cultural environment is shaped by larger forces.
- It is varied and changing.
- It has an impact on the learning of mathematics.
- Reduced economic power is accompanied by “de-mathematization”.

From schemes to symbolic mathematics

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- What visual representations help in the elaboration of schemes, and the development of abstract multiplicative thinking?
- The algebraic aspects of the fraction notation

Visual representations

- Variations in number line representations that need to be explored:
  - The double number line
  - The bar representation for quantity
- Operator diagrams

The double number line

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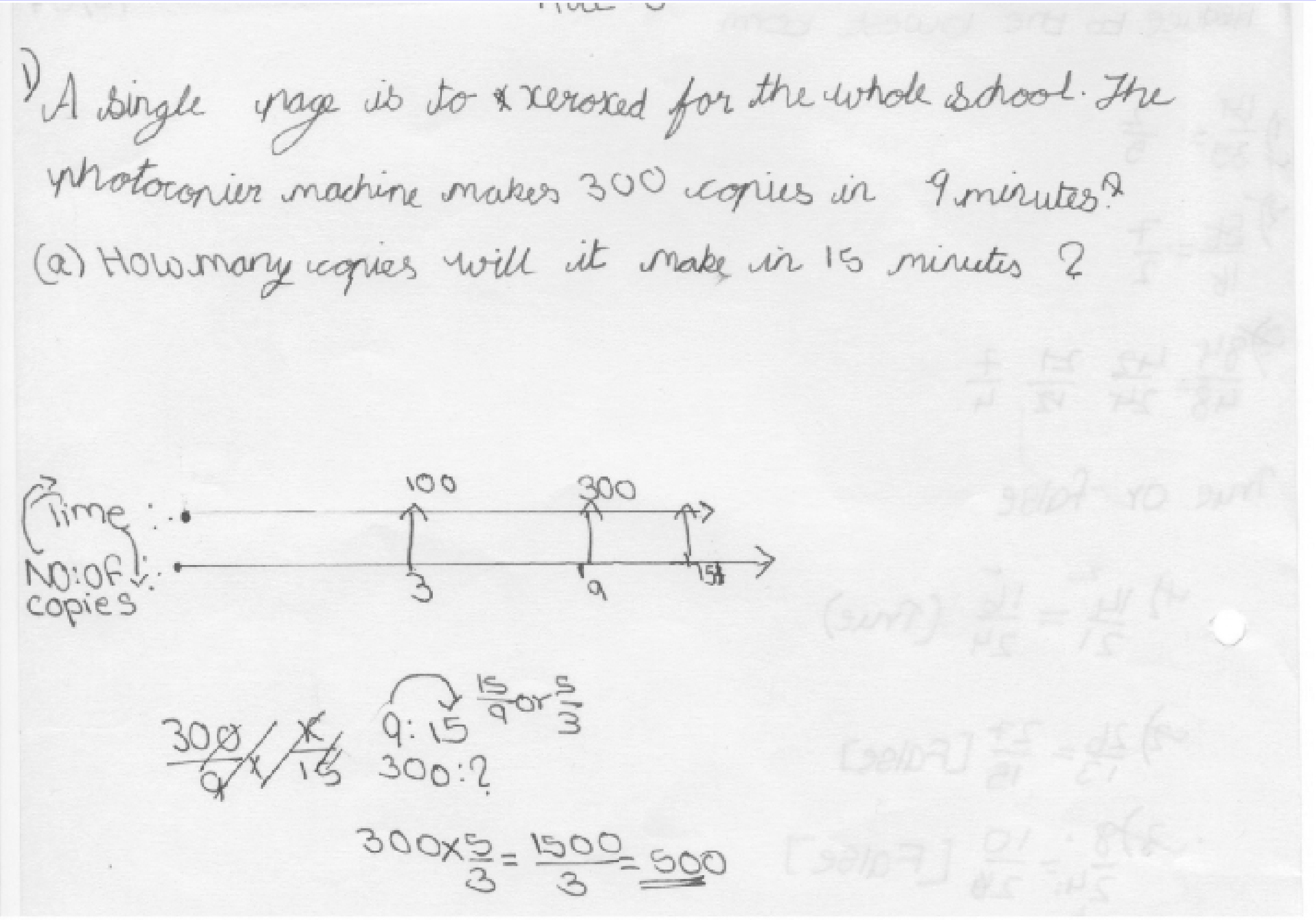
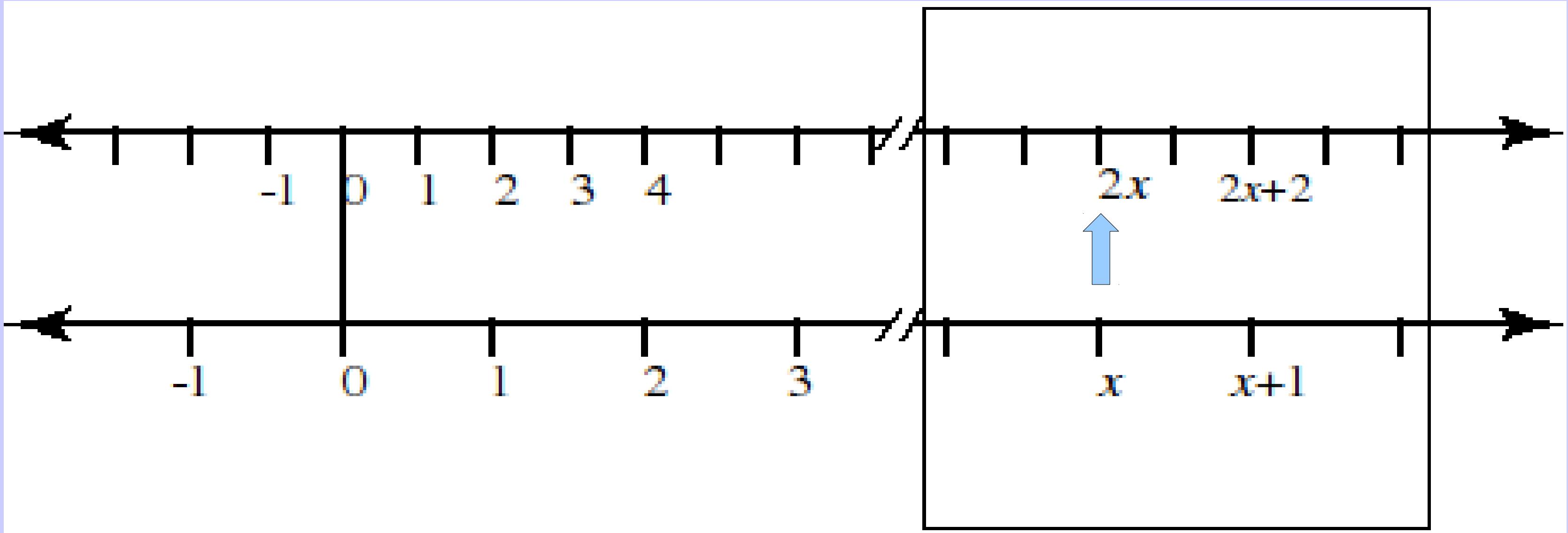
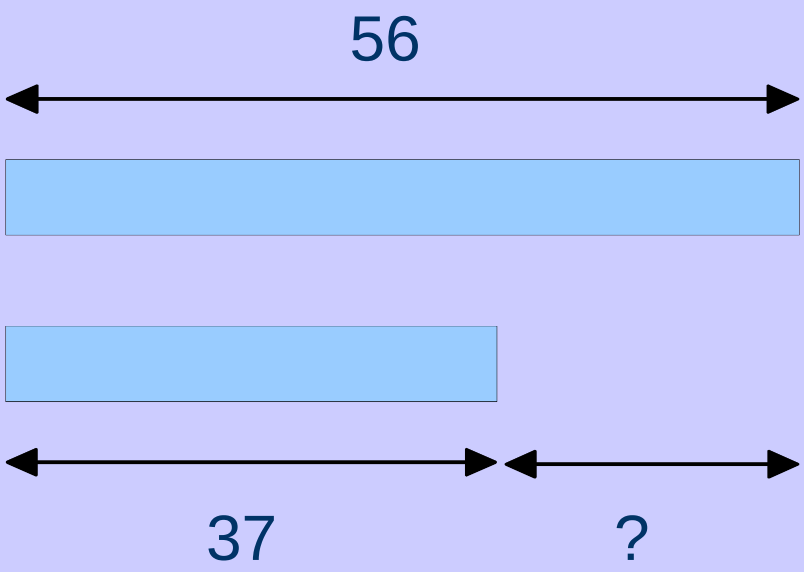

1) A single page is to ~~x~~ xeroxed for the whole school. The photocopier machine makes 300 copies in 9 minutes?
(a) How many copies will it make in 15 minutes ?

The double number line

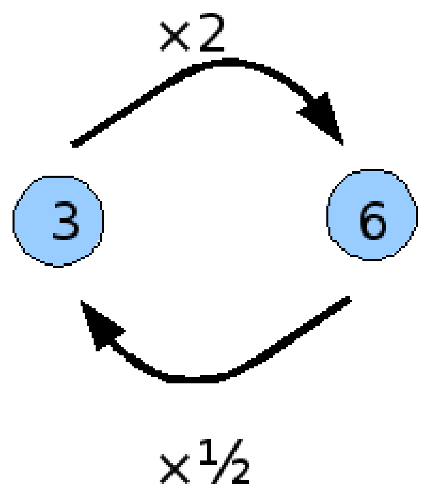


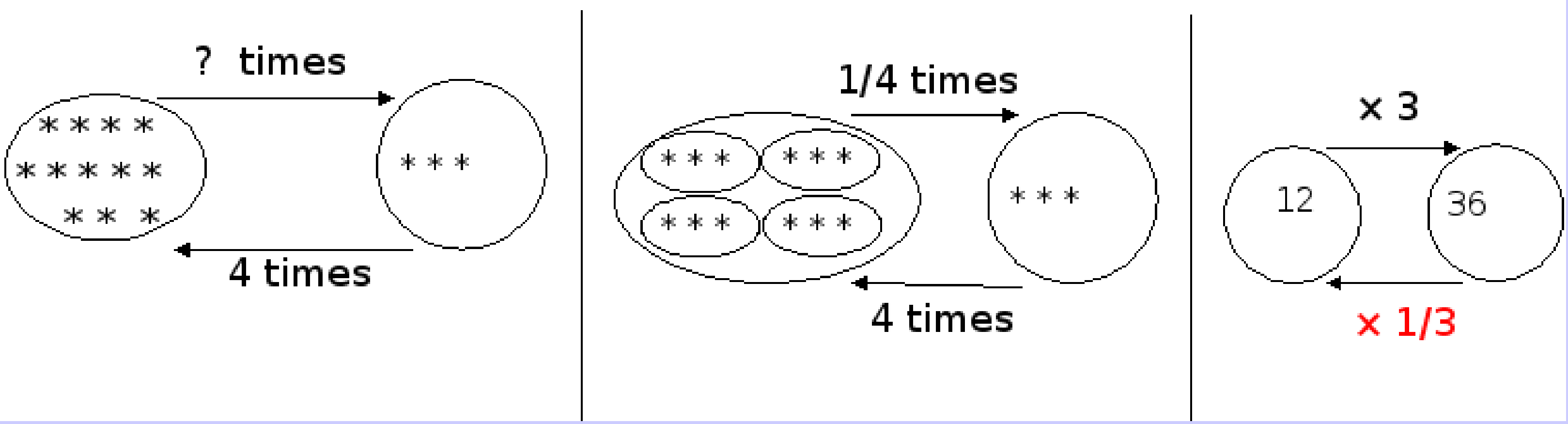
Subramaniam, 2008.

The bar representation of quantity

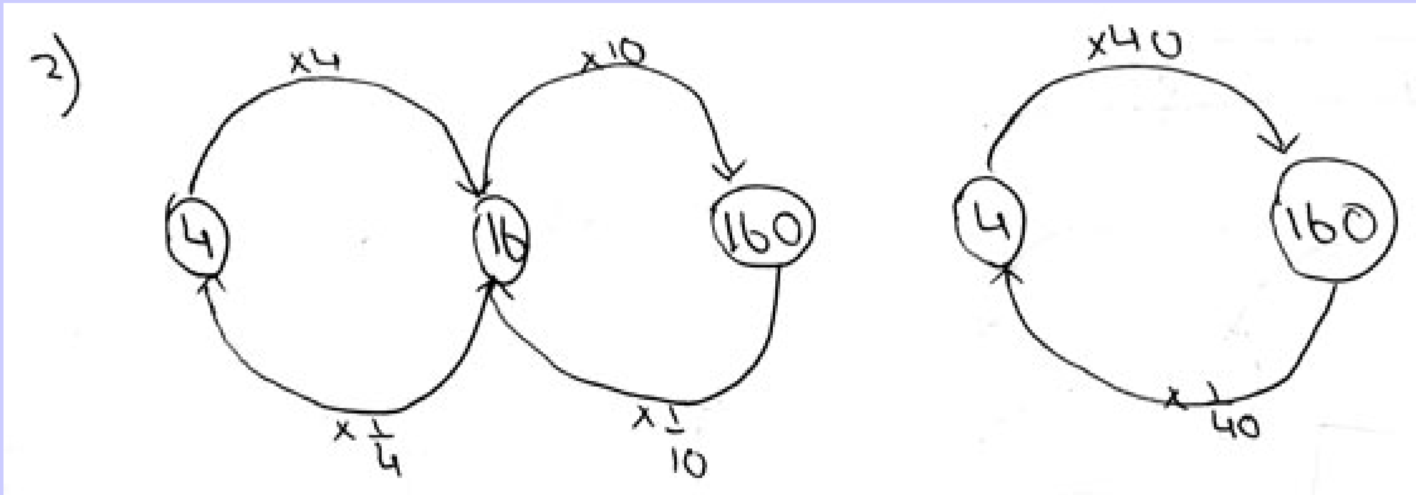


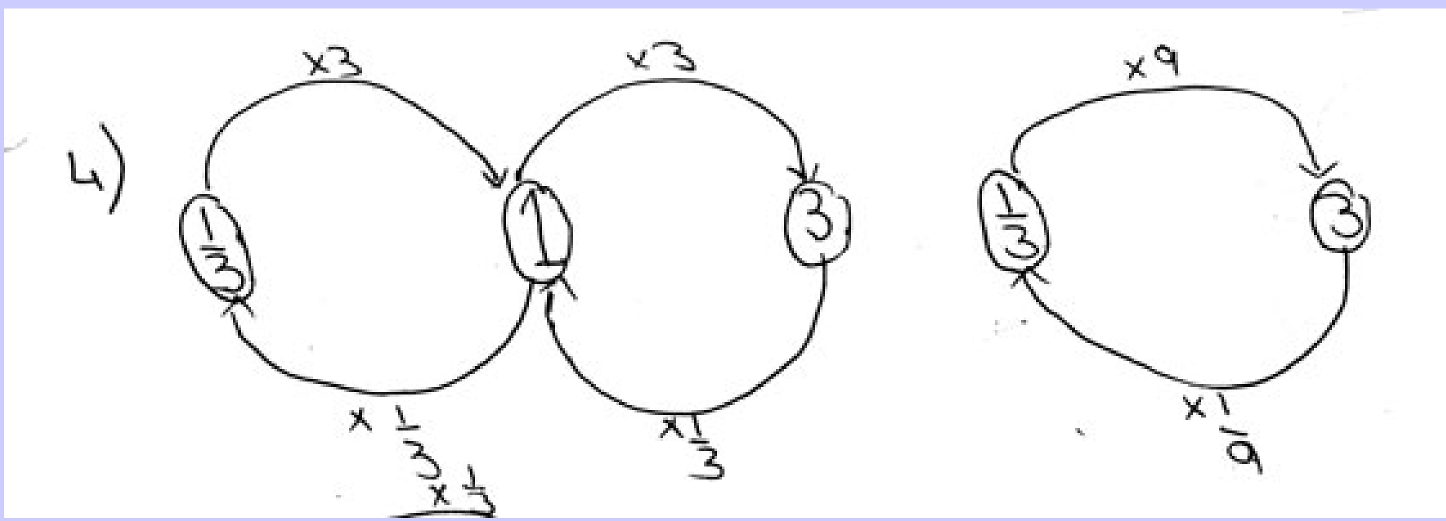
The operator diagram for multiplication





The operator diagram for multiplication





Fraction notation and algebraic thinking

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- The fraction notation has a connection with algebra:

  'x + 3' stands for the operation "add 3 to x"

  'x + 3' also stands for the result of the operation.

  'a/b' stands for the division operation a ÷ b, and also the result of the operation.

Relational thinking and fraction operations

- Relational thinking is a form of anticipative thinking in the context of operations notated in numerical expressions.
- Extensions to the fraction notation: Empson and others, forthcoming, 2011.

Numerical expressions: algebraic aspects

- Numerical (arithmetic) expressions are seen as a pre-requisite to algebra
- Usually interpreted as encoding a set of instructions to carry out operations
- Researchers have recognized the need to move from a purely "process" interpretation of expressions to a flexible "process-product" interpretation.

Numerical expressions encode operational composition

- New interpretation: numerical expressions encode the operational composition of a number.
  - 500 – 500 × 20/100
  - 5 × 100 + 3 × 10 + 6
- Attention to operational composition enhances relational thinking.

(Banerjee 2008, Subramaniam and Banerjee forthcoming, 2011)

Numerical expressions encode operational composition

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- The simplest and most common operational compositions are joint additive and multiplicative compositions.
- The fraction notation is needed to complete the representation of multiplicative composition, through the representation of division.
- So fractions need to be viewed as supporting the algebraic thinking of students.

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And many others ...

Some important references (reproduced from earlier slide)

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Thank you !