Towards an Epistemological Map of Curriculum in School Mathematics

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While being proclaimed as a move to accountability and higher standards, current educational trends in the United States, England and Australia are undoubtedly ideologically driven, influenced by underlying economic rationalist philosophies that owe much to deeper epistemological beliefs about what counts as legitimate knowledge. However existing discussions of epistemology in mathematics education tend to categorise knowledge using binaries such as qualitative or quantitative, hard or soft, pure or applied. In this paper, I describe a technique that may be useful to conduct a more nuanced analysis, and use this technique to carry out a preliminary mapping of one aspect of the proposed Australian National Curriculum in Mathematics.

Keywords: Mathematics curriculum, Epistemology

Philosophical Debates in Mathematics Education

In *The Philosophy of Mathematics Education*, Ernest (1991) emphasises the impact of philosophies of mathematics on curriculum and pedagogy. He contrasts absolutist philosophies of mathematics–logicism, formalism and Platonism with a relativist philosophy of mathematics, social constructivism. He describes how each of the absolutist philosophies have impacted on school curriculum in mathematics, through movements such as the New Math curriculum of the 1960s, the continued emphasis on ‘definition, lemma, theorem, corollary’ in tertiary undergraduate mathematics, or dichotomies between teaching by rote and teaching for understanding.

The debates and dichotomies discussed by Ernest continue today, particularly through the so-called Math Wars (Klein, 2007). Despite calls in a range of curriculum documents for a pedagogy that values the processes of mathematical investigation and problem solving and that promotes open discussion and negotiation of meaning, the majority of Western school mathematics classrooms continue to be a shifting amalgam of students learning rules and processes built on formalism, teachers striving to enable students to make connections with a broader scheme of mathematics built on Platonism, and an emphasis on logical ‘setting out’ built on logicism. Perhaps the relative failure of the problem solving and investigations movement has its roots in mismatched philosophies of mathematics, or indeed of knowledge more broadly, rather than in inadequate professional development or teacher knowledge.

The debates about curriculum, pedagogy and assessment in mathematics seem unlikely to disappear, particularly in an environment where the political imperative is largely based on raising so-called standards, commonly measured by national tests. As Luke (2003) points out:

> *Sixty years after (Dewey) the binary divide in epistemology, methodology and educational policy debates remains. Their ghosts are sustained by a persistent strain of dialectics: quantitative versus qualitative, child-centred versus behaviourist, progressive/constructivist versus direct instruction, implicit versus explicit pedagogy, project-based work versus skills orientation (Luke, 2003, p. 92).*

Thus there is an imperative to break down the binary divides and to develop a more nuanced, multi-dimensional view of knowledge production and legitimation in mathematics education. Such an endeavour may then be recontextualised into curriculum and pedagogy in schools that respects both the logic, rigour and search for truth of absolutist philosophies of mathematics and the social constructivist elements of relativistic philosophies.

Epistemological Models in Education

Snow (1960) in *The Two Cultures and the Scientific Revolution* portrayed the artistic culture and the scientific culture as diametrically opposed, making a plea for both sides to better understand each other. Snow drew particularly on his discussions with G. H. Hardy, a respected mathematician who was instrumental in focusing attention on the distinctive nature of knowing in mathematics (Hardy, 1940). Snow’s analysis was essentially one-dimensional, highlighting the differences between a scientific way of looking at the world, which valued reason and logic, with an artistic, which valued intuition and creativity.

Other analyses of knowledge have been two-dimensional. Becher and Trowler (2001), locate various fields of knowledge as either hard/soft on one dimension, and pure/applied on another. Their analysis shows how status and esteem is differentially conferred upon certain types of research and upon members of certain fields, with researchers in the hard area being more highly regarded than those in the soft and those in the pure area more highly regarded than those in the applied.

Maton (2000) describes languages of knowledge legitimation by describing classification (Bernstein, 1990), or boundary strength, on the epistemic dimension and the social dimension. He describes mathematics as having a knowledge mode of legitimation as it is usually clear what counts as valid mathematics, thus there is strong classification on the epistemic dimension. However it is less important who does the mathematics, thus it is weakly classified on the social dimension. In fields such as cultural studies this is reversed, with a legitimate voice often being reserved for members of a particular cultural group, whereas knowledge in the field is permeable and shifting. Maton terms this a knower mode of legitimation. He claims that these languages of legitimation are more than mere rhetoric; rather, they "represent the basis for competing claims to limited status and material resources" (Maton, 2000, p. 149).

Dowling (1998) also uses Bernstein's (1990) notion of classification on two dimensions, content and expression, to describe four domains of practice in school mathematics.

— The esoteric domain is strongly classified with respect to both content and expression. That is, it consists of purely mathematical topics rather than real-life contexts, and within those topics the specialised language of mathematics is used.

— The public domain is weakly classified with respect to both content and expression. It focuses on everyday contexts using everyday language.

— The expressive domain is strongly classified with respect to content, but weakly classified with respect to expression. That is, it focuses on content found within mathematics, such as algebra, but expresses it in everyday language and symbols.

— The descriptive domain is strongly classified with respect to expression, but weakly classified with respect to content. That is, it recontextualises an everyday situation using mathematical terms and symbols.

Dowling (1998) shows how school textbooks position students with respect to mathematics. He shows that textbooks written for students who are classed as of lower ability focus on the public domain, effectively positioning students as dependents in the classroom. However textbooks written for higher achieving students focused more on the esoteric domain, giving students access to more challenging and abstract mathematics, positioning them as apprentices in the mathematics classroom.

Treffers and Goffree (1985) also use a two-dimensional model, discussing the notion of mathematisation, and distinguishing between vertical and horizontal mathematisation. Horizontal mathematisation makes a problem from another field accessible to mathematical formulation. Obvious examples of this include most of the field of applied mathematics, where mathematical models are used to analyse and solve real-world problems. Vertical mathematisation refers to connections within mathematics itself, providing techniques for solving mathematical problems. Like Dowling (1998), Treffers and Goffree (1985) analysed school mathematics texts using this model of knowledge classification, arguing that good texts should have strong mathematisation on both the horizontal and vertical axes. They describe four theoretic frames:

1. A mechanistic approach has weak mathematisation both horizontally and vertically. It is characterised by an emphasis on learning skills and rules without connections to other areas of mathematics or to real-world problems. It is what Skemp (1976) describes as instrumental understanding alone – knowing how, but not knowing why.

2. A structuralist approach has strong vertical mathematisation, in that it emphasises links between areas of mathematics and thus views mathematics as a coherent, structured discipline, which can be understood relationally (Skemp, 1976). However there is weak horizontal mathematisation, with little attempt to develop the skills of mathematical modelling needed to solve real-world problems.

3. An empiricist approach sees mathematics as a tool to solve problems, thus exhibiting relatively strong horizontal mathematisation. However links between areas of mathematics are de-emphasised, thus a view of mathematics as a coherent discipline in its own right may be lost.

4. In the Realistic Mathematics Education program (Treffers, 1993) the real world serves not only as an application of mathematical problem solving but also as a source of learning. Thus both horizontal and vertical mathematisation are emphasised as real world problems served as a springboard for developing and connecting new mathematical knowledge to existing knowledge.

There are obvious similarities between these four two-dimensional models, despite their origins in very different contexts and purposes. Structurally a two-dimensional model naturally divides a field into four quadrants, with one dimension expressed vertically and the other horizontally. Each of the models uses binary distinctions on the two axes to locate and label knowledge or practice in this way. In some cases this location has the effect of privileging knowledge in one quadrant over another; in others the model merely describes, without any overt attempt to assign relative worth.

It is beyond the scope of this paper to consider each of the four two-dimensional models described above. In this preliminary analysis of the draft Australian National Curriculum the mathematisation model described by Treffers and Goffree (1985) is used. This model is shown diagrammatically in Figure 1.

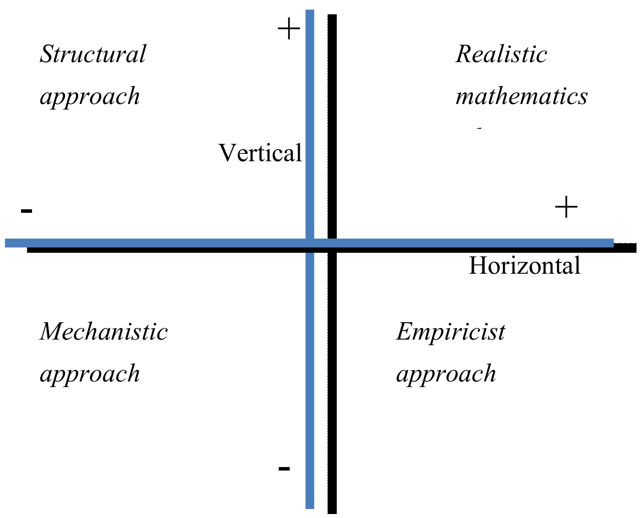

Figure 1: Horizontal and vertical mathematisation (Treffers & Goffree, 1985).

Mapping the Draft Australian Curriculum

In reality, knowledge, either in broad fields of higher education or in school mathematics, does not neatly divide into one of four quadrants. Every field of higher education has elements that are more pure than applied or vice versa, just as every field has elements of knowledge that are more towards the hard end of the scale than the soft, or vice versa. In the same way school mathematics moves around a contour of classification with respect to content and expression, and elements of the classroom may, at various times, exhibit mixes of various levels of horizontal and vertical mathematisation.

Rather than using these two-dimensional binary distinctions to locate and label practice within a particular quadrant, this paper describes a method for mapping a contour of practice according to the strength with which that practice exhibits the characteristics on each dimension. Unlike the models described above, I suggest that practice and knowledge in a field are inevitably transient, moving around a contour described by several variables, but seldom as binary opposites.

The method is developed through a preliminary analysis of the draft documents for the Australian National Curriculum in Mathematics to be implemented in schools in 2011. A national curriculum body, the Australian Curriculum, Assessment and Reporting Authority (ACARA) was set up by the Commonwealth government in 2009. ACARA was charged with developing national curriculum documents, initially in the areas of English, Science, Mathematics and History for implementation across Australia. Prior to this curriculum had been the preserve of each of the Australian States and Territories, although there had been limited agreement over a national framework for each of eight key learning areas (Australian Education Council, 1991).

The initial stage of curriculum development involved developing shaping papers for each of the four discipline areas. The papers were informed by feedback from invited experts and public consultation. The mathematics paper, the *Shape of the National Curriculum: Mathematics* (Australian Curriculum Assessment and Reporting Authority, 2009), proposes that the curriculum should focus on a limited number of big ideas at each year level in three content domains: number and algebra; statistics and probability; and measurement and geometry. It also proposes that four proficiency standards, understanding, fluency, reasoning and problem solving be embedded throughout the curriculum.

The shaping paper was then used to develop more detailed curriculum statements, initially at each of years Kindergarten to 10, to later be extended to year 12. At the time of writing the draft mathematics curriculum for K-10 is available for public consultation (Australian Curriculum Assessment and Reporting Authority, 2010). The draft contains a preliminary section describing the aims and rationale for school mathematics, the organisation of the document including a brief outline of each of the content and proficiency strands, a discussion of the characteristics of students at three broad levels of schooling, and generic capabilities to be developed through the study of mathematics. It then provides between three and eight content descriptions for each content strand at each year level. Each content description is supplemented by elaborations that suggest more specific understandings or activities embodied in the broader descriptions.

Analysis

In this paper, I analyse excerpts from the draft consultation version of the Australian Curriculum: Mathematics (Australian Curriculum Assessment and Reporting Authority, 2010) using the model of Treffers and Goffree (1985) discussed above. In order to gain an overview of the level of horizontal and vertical mathematisation, I analyse each of the content descriptions and elaborations in the Measurement and Geometry strand of the draft curriculum document for the year levels 3, 6 and 9. The Measurement and Geometry strand was chosen as one which may be likely to contain statements relating to both mathematics as an activity in its own right and as a tool for solving problems in the real world. Years 3, 6 and 9 were chosen to show the possible change and development across time.

Each elaboration of each content description was categorised according to whether the elaboration concerned predominantly horizontal or vertical mathematisation. Those concerned with horizontal mathematisation contained phrases such as “in the environment”, described actions or situations in real life, or referred to the use of real objects. Those concerned primarily with vertical mathematisation contained terms such as “equivalent ways of…”, or referred to laying the foundations for further mathematical study or understanding a mathematical concept. Each elaboration was further analysed according to the strength of that mathematisation, partly informed by the verbs in the statement. Verbs such as model, make connections, establish validity, prove or reason were rated as strong verbs as they refer to high levels of mathematical reasoning. Verbs such as calculate, measure, say or recognise were rated as weak verbs as they do not necessarily imply high levels of mathematical reasoning. A numerical value of 0 (absent), 1 (weak), 2 (moderate) or 3 (strong) was then assigned to each elaboration, and these values were then used to create an ordered pair of (horizontal mathematisation, vertical mathematisation) average strength for each content description.

For example the content description for Measurement in year 6 reads: “Solve problems involving comparisons of length, area, volume and other attributes using appropriate tools, scales and metric units”. There are four elaborations:

1. Understanding that identifying the measurement attributes that are involved in a problem are necessary before choosing the tools and units in the solution.
2. Solving problems involving comparisons of length and area, such as investigating areas of rectangles that have the same perimeter and deciding that the shape with the greatest area would be a square, or saying that when the side lengths of squares increase by 1 cm, the perimeters increase by 4 cm each time but the areas grow by increasingly larger numbers of square centimetres each time.
3. Choosing the appropriate tools and units to use in solving problems dependent on the level of accuracy required and the context of the problem.
4. Solving problems involving comparisons of lengths and volumes, such as predicting the effect of increasing or decreasing the side length of a cube on the volume of the cube or interpreting realistic situations including the volume of concrete in a pathway given that the length and width are in metres and the depth is given in centimetres. (Australian Curriculum Assessment and Reporting Authority, 2010, p. 39).

Elaboration 1 was classified as being primarily concerned with horizontal mathematisation, as there is an implication that the problem is a real life problem in which the attribute to be examined is not made explicit. As the elaboration includes the verb “understand” this was classified as relatively strong horizontal mathematisation. Elaboration 1 was thus given a rating of 3 for horizontal mathematisation. Elaboration 2 was classified as being primarily concerned with vertical mathematisation as there is no suggestion that the activity is applied to real life. It has a clear focus on making mathematical conclusions from an investigation of a mathematical context. It was classified as having strong mathematisation as the aim is to make generalisations about areas and perimeters of squares and rectangles. Elaboration 2 was rated 3 for vertical mathematisation. Elaboration 3 was classified as being primarily concerned with horizontal mathematisation as it refers to the use of tools and level of accuracy in a context. The verbs “choose” and “use” are relatively weak verbs mathematically as they do not imply high levels of mathematical reasoning. Hence elaboration 3 was rated as showing weak horizontal mathematisation, and rated as 1. Elaboration 4 contains elements of both vertical and horizontal mathematisation. The first section refers to effects on the volume of a cube when side lengths are changed, while the second section refers to a specific context. Thus it was split into two, with each section being classified separately. The first section has moderate vertical mathematisation as verbs such as “solve” and “predict” require students to reason mathematically, however there is no suggestion that students generalise or prove. The second section was classified as having weak horizontal mathematisation as it refers to a calculation where key measurements are given and the attributes identified. Elaboration 4 was thus rated as 2 for vertical and 1 for horizontal mathematisation. The mean strength of horizontal and vertical mathematisation for the content description was therefore 1.25 for both horizontal and vertical mathematisation. This was then plotted as a point on coordinate axes.

There is no suggestion that this is a rigorous technique, nor that the values assigned have any significance other than to represent a relatively fluid measure of strength. Nor has the data obtained from this analysis been triangulated by the use of an independent rater. Rather the purpose of this paper is to illustrate the possibility of conducting such an analysis and to suggest its possible extension to the other two dimensional models discussed above.

Results and Discussion

The ordered pairs obtained for each of the content descriptions in years 3, 6 and 9 in the Measurement and Geometry strand of the curriculum are plotted in Figure 2.

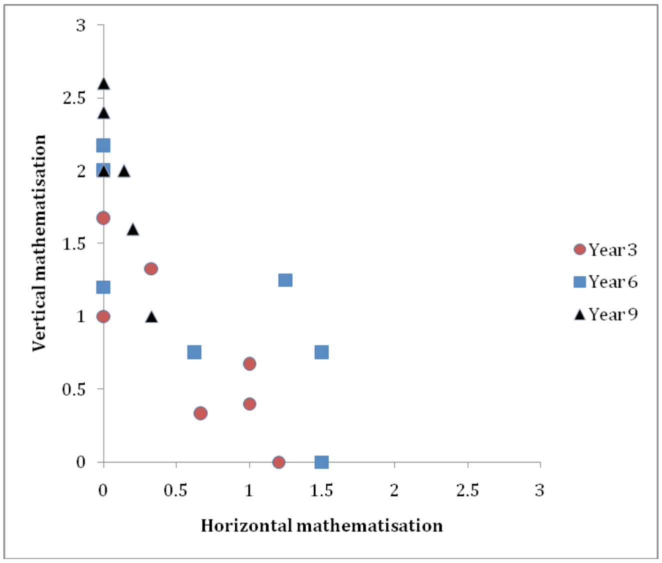

Figure 2: Strength of horizontal and vertical mathematisation in years 3, 6 and 9

This preliminary analysis suggests that knowledge does not divide neatly into one of four quadrants. Rather, the emphases in school mathematics shift around a contour of everyday and mathematical contexts, and of higher and lower levels of mathematisation. Further it suggests some trends in the level of mathematisation in years 3, 6 and 9 of the curriculum document. Firstly it would appear that the level of vertical mathematisation increases from years 3 to 9. This may well be expected as mathematical concepts become more sophisticated, and hence a stronger emphasis on making connections, reasoning and deduction might be expected. Nevertheless it may also suggest that it is important to provide greater opportunities for students in younger years to begin to develop a sense of mathematics as a connected discipline and the specific aspects of reasoning associated with working mathematically. Secondly the level of horizontal mathematisation is generally weaker in year 9 than in years 3 and 6. Indeed most of the content descriptions exhibit little or no horizontal mathematisation. This is surprising, particularly as measurement is a component of the content strand, and could be expected to be applied to real life problems. It would be interesting to conduct a similar analysis for the remaining two strands and to extend the analysis to all year levels. It is likely that there would be much greater levels of horizontal mathe-matisation evident in the Statistics and Probability strand, and perhaps greater levels of vertical mathematisation in the Number and Algebra strand.

Conclusions

The preliminary analysis has revealed some trends in the horizontal and vertical mathematisation across three year levels in one content strand of the draft Australian National Curriculum in mathematics. However it is by no means intended to be an exhaustive or definitive analysis of a curriculum document. Rather, it is meant to be illustrative of a technique that may be suitable for creating a two-dimensional mapping of epistemology in mathematics education. The technique can be extended to other two-dimensional models such as those described by Becher and Trowler (2001), Maton (2000) and Dowling (1998).

Given the current push for measurable outcomes of national testing, coupled with a call for a curriculum that emphasises fundamental skills of what, in Australia, is termed "numeracy", it seems timely to revisit epistemological and philosophical issues in mathematics education. This paper suggests that it may be possible to break down the relatively rigid classifications described to date, and to develop a more nuanced, multi-dimensional view of knowledge production and legitimation in mathematics education.

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