STUDENTS' UNDERSTANDING OF IRRATIONAL NUMBERS

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In this paper we report on our research into factors which contribute to understanding of irrational numbers. In particular, we study the influence of approximate arithmetic in the learning of irrational numbers. We also study the cognitive tools used by students to prove the infinitude of irrationals when the students are not exposed to the formal proof in the classroom. We conclude that the classroom practice of approximating numbers serves as an obstacle in distinguishing between an irrational and its rational approximation. In spite of their difficulties, we observe many students in our sample are able to provide a list of an infinity of irrationals.

Keywords: Irrational numbers, Approximation, Decimal representation, Conflict, Infinitude

INTRODUCTION

According to Fischbein, Jehiam and Cohen (1995), the idea of mathematics as a coherent, structurally organised body of knowledge is not systematically conveyed to the students and in order to do so, the emphasis must be on the logical, coherent nature of the number system. Irrational numbers constitute an important part of this number system. The research of Fischbein and others had aimed at determining how high school students and pre-service teachers comprehend irrational numbers. Based on their study they conclude that the concept of irrationals is totally confused in the minds of many students. The term *irrational* is confused with non-whole numbers, with numbers having infinity of decimals, sometimes with negative numbers, etc. Students in general are not aware of the distinction between recurring and non-recurring decimals. They also argue that students in general do not exhibit surprise in learning that segments may be incommensurable and in a given interval there is an infinity of rationals and also an infinity of irrationals. The other difficulties mentioned in the literature include providing appropriate definitions for irrational numbers (Tirosh, Fischbein, Graeber, & Wilson, 1998) and in flexible use of representations (Peled & Hershkovitz, 1999). Research done by Zarkis and Sirotic (2004) indicates that in the conception of irrational numbers and their representations, strong reliance seems to be on decimal representation of irrational numbers. They also argue that students in general do not appreciate the equivalence of the two representations of irrationals, one as that of infinite non-repeating decimal representation and the other as those not having an expression of the form a/b where a,b are integers with $b>0$. In their subsequent paper (Zarkis & Sirotic, 2005) the authors opine that the notion of irrational numbers is well understood and becomes an encapsulated object on encountering its *geometrical* representation. While the emphasis on decimal representation does not contribute to the conceptual understanding of irrationality, exposing students to the geometrical representation makes them more sensitive to the distinction between the irrational number and its rational approximation. Their experiment also reveals the difficulty of students in distinguishing between π and its rational approximations.

The essence of the research that has been discussed so far reveals the extent of the students' imagery built on their previous experiences and their difficulty in coping with the definition of irrational numbers. What few of these papers have taken into account is the full nature of students' previous learning experiences with arithmetic using finite approximations. We believe that this experience can give rise to epistemological obstacles relating to the need to simplify the new ideas in a way which fits with the students' experience.

THEORETICAL PERSPECTIVE

Concept-image and concept-definition: In their foundational work, Vinner and Tall (1981) have provided a framework for understanding how one comprehends and uses a mathematical definition.

According to Vinner and Tall (1981), to each mathematical concept, a *concept-definition* and a *concept-image* are associated. Concept image is the total cognitive structure associated with the mathematical concept in the individual's mind. Depending on the context, different parts of the concept image may get activated; the part that is activated is referred to as the *evoked concept image.* The form of words that is used to describe the concept image is called the *concept definition*. This could be formal and given to the individual as a part of a formal theory or it may be a personal definition invented by an individual describing his concept image. A *potential conflict factor* is any part of the concept image which conflicts with another part or any implication of the concept definition. Factors in different formal theories can give rise to such a conflict. A *cognitive conflict* is created when two mutually conflicting factors are evoked simultaneously in the mind of an individual. The potential conflict may not become a cognitive conflict if the implications of the concept definition do not become a part of the individual's concept image. The lack of coordination between the concept image developed by an individual and the implication of the concept definition can lead to obstacles in learning.

We recall here the notion of *epistemological obstacles* introduced by Bachelard (1938). It refers to difficulties learners experience in coming to terms with certain concepts due to their intrinsic complexity. He has classified obstacles into several types according to their source such as previously existing knowledge, use of particular language, association of inappropriate images, obstacles arising from familiar techniques and actions, and so on.

COGNITIVE PERSPECTIVE

Lakoff and Núñez (2000) use *measuring stick metaphor* to explain the birth of irrational numbers (p. 68–71). The measuring stick metaphor allows one to form physical segments of particular numerical lengths. This metaphor states that numbers are physical segments and it is possible to characterize all rational numbers in terms of physical segments. Numbers are conceived as one forms a conceptual blend–called the *Number-Physical* blend–of physical segments and numbers, constrained by this metaphor. The idea that there is one-to-one correspondence between physical segments and numbers is based on this conceptual blend. Starting with a fixed unit length, it follows that for every physical segment, there is a number. According to these authors, Eudoxus observed in 370 BCE that corresponding to the hypotenuse of a triangle with sides each equal to one unit, there must be a number $\sqrt{2}$, implicitly using the Number-Physical segment blend. This conclusion could not be achieved by using numbers *by themselves* literally. If rationals alone are believed to exist, then it would mean that the square root of 2 does not exist. But according to the Number-Physical segment blend there must be a number corresponding to the length of every physical segment and hence $\sqrt{2}$ must exist as a *number*. Thus the irrational numbers came into being.

RESEARCH

Aim

The aim of our experiment is two-fold.

1. To understand how students studying irrational numbers in the Indian scenario conceptualize the distinction between the rational representation and the actual value of π.
2. We also study the ability of students to comprehend the infinitude of irrational numbers.

Participants in our study

Our sample for this study comprised of ninety students. They were all students at a school in Chennai, studying in Grade-IX under the Central Board of Secondary Education (CBSE), India. The concept of irrational numbers was introduced in terms of decimal representation. They were taught that the set of all rationals and irrationals together make up the real number system, that a real number is either rational or irrational but not both, that a real number is represented by a unique point on the number line and conversely. Geometrical representation of some irrationals (such as $\sqrt{2}$) were given. The decimal expansions of real numbers are used to distinguish between rationals and irrationals. They are also taught to perform simple operations involving surds. The students were also taught that 1.414 is an approximate value for $\sqrt{2}$ and similarly 3.14 and 22/7 are approximations of p .

Research method

Our experiment consisted of three written tasks. Before the tasks were administered, the researcher met the course instructors and had a detailed discussion on the course content, the teaching methods adopted in the classroom. This helped the researcher to gain accurate information regarding the definitions used and which statements were merely stated as properties etc. For example, the proof of irrationality of $\sqrt{2}$ was given in the class while the infinitude of irrationals and rationals were taken as shared.

Data collection and analysis

Our data analysis broadly fits into the ideas of grounded theory developed by Glaser and Strauss (1967). More specifically, we applied open coding, which involves identifying and categorizing. All answer scripts were taken into consideration for categorizing and for the purposes of computing percentage of responses falling into each category.

The test items

1. (Task 1) Write down the definition of an irrational number in your own words.
2. (Task 2) Classify the following as rational or irrational and justify your choice.

   (i) $\sqrt{8}$, (ii) $\sqrt{81}$, (iii) π , (iv) 22/7 and, (v) 32 1/5.
3. (Task 3) Prove that there are infinitely many irrationals.

FINDINGS

Task 1- The responses of all the students for all the questions were coded and analysed. Some of the sample responses are given below. Some students have attempted to just negate the definition of a rational number to define an irrational. It shows that even while they make mistakes in writing down the definition of an irrational, they do know that if a number is not rational then it must be irrational. It shows the difficulty in formulating negation of a statement involving one or more quantifiers.

Response 1 *An irrational number is a number of the form p/q where p and q are not integers and q is not equal to zero.*

Response 2 *Irrationals when divided do not come to a conclusion but continues on and on. Example:* $\sqrt{2}$ *=1.414….... and 0.1101010000111........*

This student has failed to distinguish between recurring and non-recurring decimal expansion in his definition, however his example seems to suggest that he means non-recurring decimal expansions.

Response 3 *Irrational numbers are defined as numbers which do not include a perfect square. Example: square root, cube root, etc.*

Response 4 *A number which is neither terminating nor recurring is an irrational. It is a square root of an non-square number.*

Response 5 $\sqrt{8}$ *is irrational but* $\sqrt{8}$ *= 2.85 is rational.*

This student seems to associate the property of a number being rational or irrational to its representation rather than to the number itself. This student also fails to distinguish between a number and its decimal approximation.

Response 6 *Neither terminating nor recurring, The result of* $\sqrt{2}$ *is interpreted approximately. It is either 1.414 or 1.4145156.*

Response 7 *The radical and the radicant play an important role in determining if the form is rational or irrational.*

The varied responses are suggestive of different aspects of the definition constituting the personalized concept images.

Description	**No. of students**
Neither terminating nor recurring	41
It is not of the form p/q	32
Using examples to define	2
That which does not include a perfect square	11
others	4

Table 1: Description of categories based on students' definition of irrational numbers

Sl. No.	**Number**	**"rational"**	**"irrational"**	**other answers**
1	$\sqrt{8}$	2	88	–
2	32 1/5	84	6	–
3	$\sqrt{81}$	88	2	–
4	π	24	41	25
5	22/7	37	39	4

Table 2: Description of correct answers for Task-2

Sample responses for Task-2

Response 1 *π = 22/7 = 3.14……. Both and 22/7 are irrational.*

Response 2 *22/7 is irrational because the value goes on. There is no end digit or last digit.*

Response 3 *π has two values. When it is 22/7 it is irrational. When it is 3.14 it is rational.*

Interview

Interviewer: *How do you classify whether a given number is rational or irrational?*

Student*: I don't know. If it is of the form p/q then it is rational. But if it goes on when divided, I am not sure whether it will end or not. So it is difficult for me to say...*

Interviewer: *Why have you classified π and 22/7 as irrational?*

Student: *Because I remember π is irrational from my math class and 22/7 is taken for π to do sums both in math and physics class. So it is also irrational.*

Sample responses for Task-3

Response 1 *For example consider the irrational number 1.2834615........... Add 1 to it. It becomes irrational. If we do like this we can have an infinite number of irrationals.*

Response 2 *An irrational number is a number which is a square root of a non-square number or can be the cube root of a non-cube number and so on. If we take such numbers we will get infinitely many irrationals as these roots are different for any number.*

Response 3 *There are infinitely many irrationals. Let us take a number which is not a perfect square. Take* $\sqrt{2}$ *. Its value can differ. It can be 1.4, 1.414 and so on.*

Response 4 *Irrationals are everywhere on the real line. A line has infinitely many points. Hence there are infinitely many irrationals.*

Response 5 *Consider* $\sqrt{2}$ *,* $2\sqrt{2}$ *,* $3\sqrt{2}$ *and so on. If we do like this we will get infinitely many irrationals.*

Response 6 *Any natural number can be made into an irrational number by taking its square root, cube root, fourth root and so on. This can be done infinitely.*

Discussion

Of 90 students, 76 gave the mathematically correct definition for irrational numbers. Eighty-eight students identified "$\sqrt{8}$ as irrational and 32 1/5 as rational. This indicates that students of the sample are in general comfortable with routine calculations like simplifying $\sqrt{8} = 2\sqrt{2}$ and 32 1/5 =2 which they had carried out before concluding whether they were rational or irrational. They are also able to identify positive integers as rational. Twenty-four students classified π as rational number while 39 students classified 22/7 as irrational number. Their responses indicate that these students had approximated either π as 3.14 and concluded it to be rational or took 22/7 as π and wrote it as irrational. Many students (as can be seen from their sample responses) seem to equate π with 22/7 and 3.14. In other words they treat all the three to be equivalent.

Many students had equated π as 3.14 and concluded it to be rational (See Responses). This could be attributed to the erroneous practice of substituting π as 3.14 in computations involving areas and volumes of solid figures. Those students who had concluded 22/7 to be irrational knew π to be irrational but failed to appreciate the difference between its actual value and approximate representation. This can also be due to the practice of approximate arithmetic which they are used to, both inside and outside the classroom. According to these students the difference between the actual and approximate value is close to zero and hence the difference may be ignored. The potential conflict has not turned into a cognitive one as the definition of a rational number has not been evoked simultaneously on encountering 22/7. These errors seem natural and understandable since the students are in their early stages of assimilation of the concept of the real number system. Eighteen students had given mathematically correct proof for Task-3 in which the students were asked to show that the set of irrational numbers is infinite. These students had conceived ways generating irrational numbers proving that the set of irrationals is infinite. It appears to be quite a remarkable achievement!

According to Lakoff and Núñez (2000), the notion of infinity is conceptualized through what they call as 'basic metaphor of infinity'. For example, to get the set of natural numbers you need to collect up each number as it is formed in each iteration. The set grows without an end. It is through this metaphor that one conceptualizes an end to a never ending process. In our experiment, the proof for the infinitude of irrationals was neither taught nor discussed in the class room. But most of those who successfully proved this were able to construct the potentially infinite sequence of irrationals as they were intuitively able to conceive of potentially infinite processes. Even among the students who had given the mathematically correct proof for the infinitude of irrationals, about 60 percent of the students had written that 22/7 is an irrational number. This further reinforces the impact of the class-room practice of taking the approximate value of π as 22/7 (or as 3.14) in computations, on their understanding of irrational numbers. We also recall here the epistemological stance of constructivism in mathematics education (Von Glaserfeld, 1983). According to this model, mathematics learning is a personal knowledge construction process in which the learner seeks to assign meaning to mathematical entities. It denies objectivity to mathematical structures and claims that acquisition of mathematical knowledge takes place by constant negotiation and interaction within the community. Our finding regarding the influence of approximate arithmetic on the students' understanding of the notion of irrational numbers seems to be consistent with this view. Or it could be a result of lack of coordination among the various concept-images of newly acquired knowledge components.

References

Bachelard, G. (1938), (Reprinted 1980). *La Formation de l'esprit scientifique*. J. Vrin, Paris.

Fischbein, E., Jehiam, R., & Cohen, C. (1995). The concept of irrational numbers in high school student and prospective teachers. *Educational Studies in Mathematics*, *29*, 29-44.

Glaser, B.G., & Strauss, A. L. (1967). *The discovery of grounded theory: Strategies for qualitative research*. New York: Aldine de Gruyter.

Lakoff, G., & Núñez, R.E. (2000). *Where mathematics comes from: How the embodied mind brings mathematics into being*. New York: Basic Books.

Peled, I., & Hershkovitz, S. (1999). Difficulties in knowledge integration:revisiting Zeno's paradox with irrational numbers. *International Journal of Mathematics Education, Science and Technology, 30*(1), 39–46.

Tiorsh, D., Fischbein, E., Graeber, A. & Wilson, J. (1998). Prospective elementary teachers' conceptions of rational numbers. Retrieved from the World Wide Web.

Vinner, S., & Tall, D.O., (1981). Concept image and concept definition in mathematics with particular reference to limits and continuity. *Educational Studies in Mathematics*, 12, 161–169.

Von Glasersfeld, E. (1983). Learning as a constructive activity. *Proceedings of the fifth PME-NA*, *1*, 41–69, Montreal.

Zarkis, R., & Sirotic, N. (2004). Making sense of irrational numbers: Focusing on representations. Proceedings of *28th International Conference for Psychology of Mathematics education.* 4, 497–505. Bergen, Norway.

Zarkis, R., & Sirotic, N. (2005). Locating irrational numbers on the number line. Proceedings of the *27th Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education.*