6. Research on the Learning of Fractions and Multiplicative Reasoning: A Review

K. Subramaniam
Homi Bhabha Centre for Science Education, (TIFR), India
subra@hbcse.tifr.res.in

Introduction

The topic area of fractions is acknowledged to be difficult for school students to learn. We began working on approaches to the teaching and learning of fractions as a part of the elementary mathematics curriculum development initiative at the Homi Bhabha Centre. This led us to ask questions about why the topic of fractions is important and to ask how it is connected to other topics in school mathematics. I'll begin this review by addressing this question, and how the place of fractions in the curriculum is understood in the light of the research of the past few decades. I'll then describe the main trends of the intensive phase of research on rational number learning in the period between the mid-1970s to the mid-1990s. Extensions of this research in the subsequent period together with some new strands are addressed next. There is a need to integrate this vast body of research findings in a manner that can impact curriculum design and teaching. I end the review by sketching lines of research that can help in meeting this goal.

The reasons commonly cited to justify the inclusion of fractions in the traditional curriculum are: (i) the fraction concept forms the conceptual basis for decimals and percentages and (ii) the arithmetic of fractions is needed for algebra. These are valid reasons for giving importance to fractions in the curriculum. Elaborating on these reasons, we may say that there are three topic areas for which fractions are important. Fractions arise naturally in connection with measurement. We quantify continuous magnitudes like length by choosing a unit and we may need to subdivide the unit for greater accuracy. Fractions are needed to express and compute with measures that are smaller than the chosen unit. Secondly, fractions are useful in dealing with proportionality, in expressing ratios, in comparing and manipulating them. Finally, the fraction notation and fraction operations are needed in algebra to express quotients and to operate with them.

We note that the topic area of measurement actually requires only a subset of fractions. In modern scientific measurement contexts, only decimal fractions, notated using the decimal point, are used. In contexts from the everyday world and beyond, the use of percentage to approximate ratio is convenient because it recasts the ratio or fraction in question as a fraction with denominator 100, a salient fraction in the decimal system. Earlier times and cultures used other subsets of fractions. For example, the British subdivisions of the inch were in binary fractions (½, ¼, ⅛, etc.) and the astronomers of ancient Iraq computed with sexagesimal or base 60 fractions. The fact that measurement contexts can make do with a subset of fractions has sometimes led educators to argue for including only decimal fractions in the curriculum, and to abandon fractions in general and especially the arithmetic of fractions (Varma & Mukherjee, 1999). However, the exploration of multiplicative relationships in a variety of contexts and the use of the fraction notation in algebra requires rational numbers in general and hence fractions beyond the decimal fractions.

Why are fractions useful in the contexts discussed above and what is common to these contexts? We find that all of them involve multiplicative relationships between magnitudes, quantities and numbers. Fractions are the basic tools in dealing with such relationships. By multiplicative relationship, I mean a mathematical relation between magnitudes, quantities and numbers constituted by the multiplication operation. The division operation is included here as the inverse of the multiplication operation. A familiar kind of multiplicative relation is the ratio. Additive comparison of two magnitudes is a judgement about additive relationships: how much larger or smaller a magnitude is than another ("my brother is two inches taller than me"). Multiplicative comparison is about the multiplicative relation between magnitudes: how many times the smaller magnitude is the larger or vice versa ("I am twice as tall as my daughter"). This relation is often expressed as a ratio. Multiplicative comparison is at the heart of measurement, which is essentially answering the question "how many times the unit is the target measure?"

Situations involving proportional relationships need one to not only make multiplicative comparisons, but to grasp that certain multiplicative relations in the situation remain constant or invariant. For example, suppose one needs to use a recipe written for two persons to prepare a dish for five persons. While recalculating the ingredients in the recipe, one assumes that the ratio between the number of people and the quantity of each ingredient in the recipe is constant. This is an example of a direct proportion. The situation of inverse proportion is different: If a bag of rice is enough to feed 12 people for 15 days, how many days can 20 people be fed with the bag of rice? Here the invariant multiplicative relation is not the ratio but the product of the number of people and the number of days.

Multiplicative thinking or multiplicative reasoning (these two phrases are used interchangeably in this review) involves grasping multiplicative relationships in situations and dealing with them mathematically in appropriate ways. It means not only reasoning about quantities and their relationships but also the ability to represent the relationships mathematically and facility with generating and transforming representations. One reason for teaching fractions is that they are useful tools for solving problems requiring multiplicative reasoning. But fractions are not only tools, they also create learning opportunities. From the pedagogical point of view, the topic of fractions is important because it offers an opportunity to *develop* multiplicative reasoning. As children grasp the meaning of the fraction notation and interpret it in contexts, as they learn to compare fractions, and as they learn to apply fractions, their multiplicative reasoning develops. This is an additional pedagogical justification for including the topic of fractions in the curriculum.

Researchers who studied the teaching and learning of fractions in the 1980s recognized that fractions are an integral part of a larger connected network of topics in mathematics which includes multiplication and division, measurement, proportionality, and at higher levels includes linear functions and vector spaces. This connected network of topics has a vast vertical elaboration and has been described as the ‘multiplicative conceptual field’ (Vergnaud, 1994) and the underlying related ideas as ‘multiplicative reasoning’ (Harel & Confrey, 1994). The mathematization of many aspects of reality involves identifying and expressing linear relationships between quantities or magnitudes that are varying. In actual practice, dealing effectively with linear relationships can be complex and call for thoughtful application of mathematical ideas and tools.

Some researchers prefer to use the phrases ‘rational number’ and ‘intensive quantity’ to refer to the core constructs that underlie multiplicative reasoning. We need to clarify the use of these terms. ‘Rational number’ is unambiguously defined in mathematics as a number which can be expressed in the form p/q where p and q are integers and $q \neq 0$. In the elementary school curriculum, only sub-sets of rational numbers are first introduced to students, making it necessary to use a different term, namely ‘fractions’. The term ‘fraction’ is used to mean different things and we can distinguish a narrow and a broad sense. In the narrow interpretation, fractions refer only to the positive rational numbers. Fractions may also be interpreted broadly to refer to any real number expressed using the fraction notation and may include such numbers as $\pi/2$ (Lamon, 2007). I prefer to use the term ‘fraction’ in the broad sense. Even though the scope of this review requires only the narrow sense, it is useful to keep the broader sense in mind since it reminds us of the connection that fractions have with topics beyond the elementary mathematics curriculum. Sometimes I use the phrase ‘fraction notation’ to explicitly indicate the broad sense, and to also refer to ‘fractions’ involving variables and numerical or algebraic expressions. ‘Negative’ fractions (i.e. fractions marked by the use of the “–” sign), do not figure in this review which discusses issues concerning the teaching and learning of fractions, since negative numbers constitute, pedagogically speaking, an important and different topic domain.

The term ‘rational number’ is used widely in the literature, even though most studies discuss only positive rational numbers (i.e., fractions in the narrow sense). So I will use this term frequently in this review interchangeably with ‘fraction’. Finally, we must abandon one especially narrow interpretation of fractions. Sometimes teachers and educators hold an exclusive part-whole interpretation of fractions, which excludes even fractions greater than one as ‘improper’. Such an interpretation is neither mathematically compelling nor pedagogically fruitful.

Intensive quantities, which are contrasted with extensive quantities, are familiar from contexts in physics. Extensive quantities like mass, length, volume, time, vary with the extent of matter or ‘substance’, while intensive quantities like density, speed, pressure are ratios of extensive quantities and do not depend on how much substance there is. More generally, intensive quantity can mean any ratio. The concentration of orange juice may be expressed as a ratio of concentrate to water, an intensive quantity. Probabilities are expressed as ratios and are hence intensive quantities. It is reasonable to hypothesize that understanding intensive quantities and the numbers that denote them (ratios or rational numbers) is more difficult than understanding extensive quantities and the numbers that denote them. Nunes and Bryant (2008) make this hypothesis and suggest further that one of the chief justifications for teaching fractions or rational numbers is that they are needed not only to express but also to conceptualize intensive quantities. Ratios and intensive quantities, as I suggested earlier, are instances of the multiplicative relation between quantities. The term 'multiplicative relation' is more general and includes ratios, as well as the relation between factors and products.

To summarize, fractions have an important place in the curriculum not only for the utilitarian reasons of being useful in computation in arithmetic or algebra. Fractions are also important for conceptual reasons. The topic of fractions provides an occasion to strengthen multiplicative reasoning. Fractions extend children's concept of number, making it possible to quantify extensive as well as intensive properties and to connect numbers and measures more fully. In the following section, we will elaborate on what is meant by multiplicative reasoning and then discuss the issue of the relation between fractions and multiplicative reasoning.

The Development of Multiplicative Reasoning

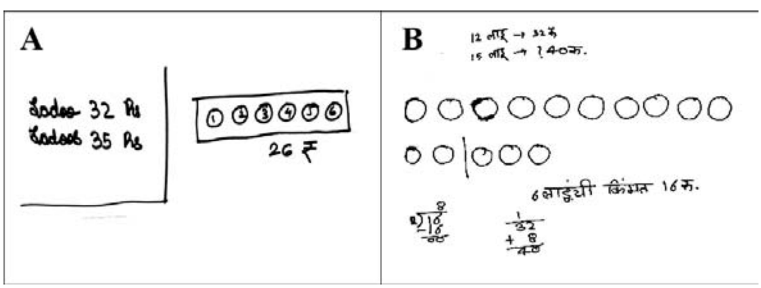
To many mathematically literate adults, proportional thinking is natural and it may seem that there is no reason to emphasize an idea that is rather obvious. But the fact that multiplicative relationships are internalized well by adults does not mean that it is easy or trivial for children. Multiplicative reasoning is cognitively demanding. As we have noted, quantification of magnitudes like length and weight also involves multiplicative thinking. Quantification of measures is a significant cognitive achievement in itself. Attending to the relation between quantities is a higher order capability. For Piaget, proportional thinking was an instance of this higher order thinking because it involved grasping the relation between relations (Piaget, 1952). Direct proportional relationships are those in which the ratio of two varying quantities remains the same – understanding direct proportion involves perceiving the equivalence (relation) between two ratios (relations). Inverse proportion situations are also instances involving the comparison of the multiplicative relation between quantities. One must comprehend the invariance of the product and the compensating multiplicative relation between the quantities that are varying.

Given the wide range of multiplicative thinking, one may expect it to develop gradually over an extended period. We can distinguish two broad phases of development. The early phase, which is marked by the development of multiplicative reasoning as an informal capacity, involves the shift from additive thinking to recognizing and dealing with simple proportional relationships.

Researchers studying the early development of multiplicative thinking interpret it as a hypothetical mental construct, underlying children's responses to proportion problems (Empson, Junk, Dominguez & Turner, 2005; Tzur et al., 2012). We illustrate this sense of multiplicative thinking with an example taken from our work with young children. Figure 1 below shows the responses from two grade 4 students to the problem: "if 12 ladoos cost Rs 32, how much will 15 ladoos cost?" Student A concludes that 15 ladoos will cost Rs 35. On prompting with a further question, he says that 6 ladoos will cost Rs 26. This is an example of what is usually called 'additive thinking'.

Student B in contrast struggles to solve the problem by trying to find the cost of one ladoo. When this proves too difficult, he is asked by the researcher if he can solve the problem without finding the cost of one ladoo. He then finds the cost of 6 ladoos as Rs 16, and the cost of 3 ladoos as Rs 8. Adding 32 and 8, he finds the cost of 15 ladoos as Rs 40. B's solution as well as the reasoning process is different from that of A. A sets up the correspondence between costs and ladoos as 'one unit for one unit': 12 ladoos cost Rs 32, 13 cost Rs 33, and so on. B sets up a multiplicative correspondence between quantities which are in multiple units: 12 ladoos cost Rs 32, so 6 ladoos (half of 12) cost Rs 16 (half of 32), and so on. Here 12 ladoos are treated collectively as a unit or, to use the more precise phrase, as a 'unit of units'. Thus one difference between additive and multiplicative thinking is that the former involves a 1-1 correspondence between singleton units, while the latter involves a 1-many or a many-many correspondence between units of units.

'Build-up' strategies of the kind displayed by student B are commonly found in the spontaneous 'unschooled' solutions to proportion problems by both children and adults (Nunes & Bryant, 2010). Such strategies indicate only the beginnings of multiplicative thinking. When the numbers are awkward, build up strategies may break down. Studies have found incorrect build up strategies which combine multiplicative and additive thinking (Misailidou & Williams, 2003). The following can serve as an example. Problem: if 12 ladoos cost Rs 32, how much do 19 ladoos cost? Attempted solution: 12 ladoos → Rs 32; 18 ladoos (12 + 6) → Rs 48 (32 + 16); hence 19 ladoos cost Rs 49. The prevalence of such strategies suggests that the further development of multiplicative thinking from an informal capacity to a mathematical ability needs the explicit use of the multiplication operation, supported by the use of multiplication facts and procedures.



Student A: Rs 32 for 12 laddoos; Rs 35 for 15 and Rs 26 for 6.

Student B: Rs 32 for 12 laddoos; Rs 16 for 6 and Rs 40 for 15.

Figure 1: **Responses of 2 students to the question: if 12 ladoos cost Rs 32, how much do 15 ladoos cost?**

Once children begin to use multiplicative strategies with assurance, the second broad phase of its development is underway. This involves the application of multiplicative thinking to diverse situations beyond those involving simple, direct proportional relations. Examples are situations of inverse proportion and multiple proportions. The latter are situations where one variable is directly proportional to two or more variables. For example, the rice ration for a group of persons varies linearly as the number of persons as well as the number of days. Other situations where the multiplicative relation is important are those where the product of measures is salient. These include situations involving geometric attributes such as area and volume and physical quantities such as moment of a force/weight. Many situations involving such attributes require the co-ordination of multiplicative strategies with spatial or physical reasoning (Rahaman, Subramaniam & Chandrasekharan, 2012). Another important development is distinguishing between situations where it is appropriate to assume proportionality and situations where it is not (De Bock, Van Dooren, Janssens & Verschaffel, 2002). Within mathematics, a variety of topic domains call for multiplicative reasoning: whole number multiplication and division, fractions, decimals, ratio, percentages, proportionality and linear functions. Further development leads to competence in using linear relationships as tools in the analysis of situations which involve complex relationships between quantities as in calculus. Thus multiplicative thinking forms the backbone for a large part of the mathematics curriculum in school and beyond.

The examples of informal proportional reasoning that we have seen involve partitioning and chunking quantities in appropriate and convenient ways. The learning of fractions brings forth many opportunities to develop facility in partitioning and convenient grouping. Further, fractions greatly expand the range of proportional situations that can be handled, including those where the whole numbers involved are not multiples of other numbers and where the quantities themselves are in fractional units. Fractions also are a handy notation to express ratios and multiplicative relations in general. Some of the themes of research in the period discussed below offer ways of analysing the basic constructs underlying fractions and multiplicative reasoning.

Research on the Learning of Fractions and Multiplicative Reasoning from the Mid-1970s to the Mid-1990s

The fact that fractions is a difficult topic for students to learn has been recognized for long. Piaget's studies had a major role in resetting the agenda of researchers who were studying children's difficulties in learning fractions. Research prior to the influence of Piagetian constructivist psychology focused on the details of student errors in implementing computational algorithms in fraction arithmetic, on the hierarchy of skills needed for the arithmetic of fractions, on how to teach algorithms and on how fraction manipulatives can help (Novillis, 1976; Payne, 1976). Piagetian ideas began to shape mathematics education research in the 1960s and early 1970s, first in the topic domain of whole number learning and then in the learning of rational numbers and other topics (Steffe & Kieren, 1994). The period from the mid 1970s to the mid 1990s witnessed an intensive phase of research on rational numbers mainly by researchers in North America and Western Europe. In this section, I'll summarize the research done during this phase under four main themes followed by remarks about how these themes and ideas have continued to influence later research.

Fraction Subconstructs

The impact of Piagetian studies on mathematics education together with the influence of the earlier school of gestalt psychologists, led researchers to focus on issues of meaning and understanding in learning mathematics. Thomas Kieren (1976, 1988) offered an analysis of the fraction concept that went beyond symbolic and computational aspects. Focusing on how fractions are interpreted in diverse situations, he argued that the fraction concept is composed of several sub-constructs. The idea of sub-construct was taken from the philosopher of science Henry Margenau, who made a distinction between fact and mental construct. Facts belonged to the level of objects in the real world. Constructs, which were furnished by the mind, belonged to a hierarchy of levels increasingly removed from the world. Mathematical constructs were distant from the real world. Intermediate between the mathematical construct of rational number, and facts in the world, there were sub-constructs that interpreted fractions in terms of a real world context or situation.

Kieren distinguished five sub-constructs of fractions: part-whole, measure, ratio, quotient and operator. The *part-whole* subconstruct is familiar from standard area diagrams in textbooks where the fraction indicates a shaded part of a whole. Most diagrams show both the shaded and the total area as composed of equal sized parts, although explicitly showing this is not necessary. The *measure* subconstruct is exemplified in contexts where measures are denoted by fractions: 3½ kg, 1.25 m, half an hour, etc. The representation closest to the measure subconstruct is the numberline, where fractions smaller than or larger than 1 can be represented. The *ratio* sub-construct arises in situations involving not only part-whole but also part-part comparisons such as the ratio of boys to girls in a class. Ratios may also encode comparisons of magnitudes not related as parts or wholes, for example, in comparing two adjacent sides a rectangle. Ratios may also encode comparisons of quantities belonging to different measure spaces (e.g. two spoons of sugar per cup).

The *quotient* sub-construct encodes the result of the division operation. A familiar context embodying this interpretation is the situation of equal sharing. When 2 rotis are shared equally among 3 children, the share of each child is 2 ÷ 3 = 2/3 roti. For many children, it is surprising that when *m* units are divided equally among *n* persons, the share of each person is simply *m/n* unit. The *operator* sub-construct has the sense of 'a fraction *of* a certain quantity', as for example, when we say that we have covered 2/3 of a total distance of 60 km or that ¾ of a 500 ml packet of ice-cream is over.

Although the sub-constructs are all derived from the mathematical construct of rational number, they have slightly different meanings, are connected to different contexts, and have distinct meanings and possibilities for the various operations with fractions. Addition is easy to interpret in terms of the measure sub-construct, but is difficult in the case of the ratio sub-construct. Multiplication is easy to interpret in terms of the operator or ratio sub-construct. Fractions greater than one are difficult to interpret strictly within the part-whole construct.

Kieren proposed that the difficulty of learning fractions was due in part to the multiple interpretations of fractions that students had to internalize. The curriculum did not help much since it typically focused only on one or two of these interpretations. Kieren's suggestion was to use a variety of contexts to allow for the development of the whole range of sub-constructs. In later publications, Kieren developed an integrated framework for progression from fraction sub-constructs to other multiplicative concepts leading on to the formal understanding of rational numbers as a quotient field (Kieren, 1993). Kieren also developed test items to assess students' understanding of fractions. His finding that students tend to perform differently on items related to the different sub-constructs provided evidence for distinguishing between the sub-constructs. The sub-construct theory had an impact on other researchers who explored the semantics of the concept of fractions, and developed an analysis of how students' reasoning proceeded with regard to the different subconstructs. We shall discuss the semantics of partitioning and unitizing in a later subsection.

Semantics of the Multiplication Operation

In the 1980s, not only fractions, but other core topics in the middle school mathematics curriculum became central topics of research. Several researchers focused on students' difficulties with the multiplication operation. As students move to the middle school, the multiplication operation becomes more prominent than the addition operation and is extended to new contexts calling for new interpretations. This presents challenges to students who have by then developed robust conceptions of whole numbers and operations on them. Students are first introduced to multiplication as repeated addition, but this interpretation is limited and unhelpful in many contexts which involve proportionality or multiplication of rational numbers (Hiebert & Behr, 1988). Intuitive notions such as 'multiplication makes numbers bigger' or 'division makes smaller' developed while dealing with whole numbers become restrictive and misleading (Fischbein, Deri, Nello & Marino, 1985).

The semantics associated with the multiplication operation is significantly different from the addition and subtraction operations. Schwartz (1988) pointed out that the vast majority of situations involving the multiplication operation follow the $I \times E = E'$ pattern: an intensive quantity multiplied by an extensive quantity yields another extensive quantity as a result. The example below follows this pattern:

Cost of 5 kg of potato:

cost in Rs per kg × weight in kg = cost in Rs

32 (Rs per kg) × 5 kg = Rs 160

The first quantity, cost per kg, is an intensive quantity. The second quantity, weight, is an extensive quantity. The product is neither of these two quantities, but a third quantity, cost, which is an extensive quantity. Schwartz called multiplication a 'referent transforming' operation, since quantities are not preserved by the operation, unlike in the case of addition (rupees + rupees = rupees).

The multiplication operation is referent transforming even when the pattern is different from the one instantiated above. Consider for instance a different pattern: $E \times E = E'$. Here two extensive quantities are multiplied to yield another extensive quantity. An instance of this pattern is:

Area of a rectangle: length (cm) × breadth (cm) = area (cm^2)

In some cases, multiplication keeps one of the referent quantities being multiplied unchanged. In unit conversion situations (expressing a length given in inches in terms of cm) the quantity does not change, but the unit does. In scalar comparison or transformation ("I am twice as heavy as I should be" "Double the quantity of rice ordered!"), neither the quantity nor the unit changes.

Schwartz's analysis significantly called attention to the role of intensive quantities in situations commonly modelled by multiplication and the focus on the nature of quantities and units in the multiplication operation. Even though it did not illuminate children's difficulties in specific ways, it provided insights about what was conceptually important in teaching for developing multiplicative reasoning. It was also suggestive of ways in which curricular goals could be organized to link the development of multiplicative thinking with quantification and mathematical modelling. However, these suggestions have not been explored in a systematic manner in subsequent research studies.

Missing value proportion problems are a critical launching point for the development of the multiplicative conceptual field. These are problems where three quantities of a proportion are given and the fourth is required to be found. Vergnaud (1988) provided a comprehensive analysis of missing value proportion problems and student solution strategies arising in simple and multiple proportion situations and situations involving products of measures. Vergnaud noted the similarity between product of measures and multiple proportion situations. He also found a preference for working with scalar ratios (within measure space ratios) among students. These contributions are well recognized and described in several reviews of the field (see for example, Lamon, 2007).

Semantics of Partitioning and Unitizing

A group of researchers from several U.S. universities implemented a long term research programme on the teaching and learning of fractions called the 'Rational Number Project' from 1979 to the 1990s (See http://www.cehd.umn.edu/ci/rationalnumberproject/). Wide ranging research studies were undertaken on fraction learning as well as proportional reasoning using a variety of methodologies including the teaching experiment. The efforts of the RNP group of researchers were important in bringing together and intensifying research on rational number learning. An important theoretical contribution made by this group of researchers was the semantic analysis of the unit concept, which connected multiplicative thinking and fractions. In many instances, a given quantity may be conceptualized in terms of different units. To take a simple case: 3 *rotis* may be thought of as just 3 single *rotis* or as one packet of 3 *rotis*. In the notation used by the researchers the first case would be notated as 3 [(1-unit)s], while the second as 1[(3-unit)] (Behr, Harel, Post & Lesh, 1992). An important aspect of multiplicative thinking and fraction understanding is the ability to conceptualize the same quantity in terms of different units in a flexible manner, described by the phrase 'flexible unitization'.

Figure 2 illustrates, following Lamon (2005), progressively more complex conceptualizations of the unit. The ratio unit puts two units in correspondence: 3 units per person. Conceptualization of the ratio units lies at the heart of multiplicative thinking. In the example of children's thinking discussed earlier (see Figure 1), the successful strategy used by Student B entails the conceptualization of the ratio unit Rs 32 for 12 ladoos, and the knowledge that this is the same as the unit Rs 16 for 6 ladoos. The analysis proposed by Post et al. envisages intermediate steps such as Rs 32 for 1[12-unit] or Rs 32 for 2 [6-units], hence Rs 16 for 1[6-unit]. Their work presented detailed notations for such transformations involved in proportional and multiplicative thinking. Although, this detailed analysis has been directly used by very few researchers, the closely related idea of children's fraction schemes has informed the perspective of several researchers.

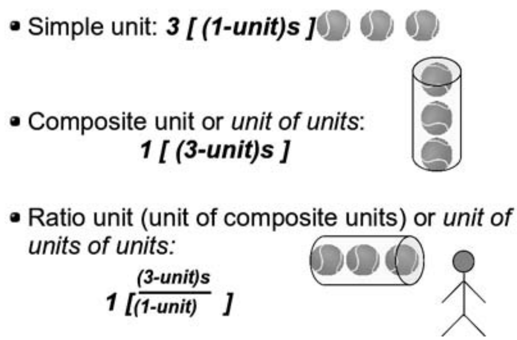


Figure 2: **Conceptualizing units of different kinds.**

The basic idea of flexible unitization however is powerful and has been developed in fruitful ways in the work of Susan Lamon (2002). Lamon's tasks combine partitioning with unitizing in ways that encourage students' reasoning. Figure 3 shows a task that is powerful in eliciting a variety of reasoning strategies from children. The shaded part of a rectangular area is unitized in different ways to give different, but equivalent fractions.

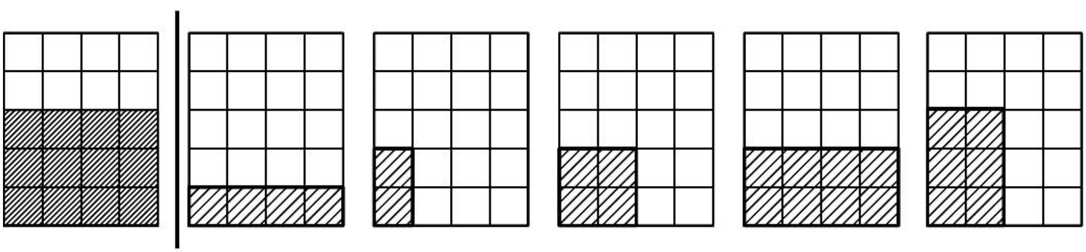


Figure 3: **Partitioning and unitizing in flexible ways: the rectangle on the left shows the fraction 12/20, which can be measured with each of the units shown in the rectangles on the right to arrive at the equivalent fractions (left to right).**

$\frac{3}{5}, \frac{6}{10}, \frac{3}{5}, \frac{1\frac{1}{2}}{2\frac{1}{2}}, \frac{2}{3\frac{1}{3}}$. **(Adapted from Lamon, 2002.)**

Exploring the Equal Sharing Context as a Starting Point for Teaching Fractions

Most textbooks present fractions initially as parts of a whole by partitioning the area of shapes such as circles and rectangles into equal parts. A context in which the act of equipartitioning arises naturally is when a whole is to be equally shared among several persons. Such contexts are close to the life experience of children. It is hence surprising that most textbooks in India, and perhaps elsewhere, do not use this context to introduce and develop the concept of a fraction. In the 1980s, the use of sharing contexts to teach fractions was explored in detail in instructional settings through the work of Streefland (1993). Streefland was a researcher at the Freudenthal Institute, where the Realistic Mathematics Education approach was developed. In terms of the fraction sub-constructs discussed above, this approach exploits the quotient sub-construct, where the fraction *p/q* is interpreted as the share of each person when *p* wholes are shared equally among *q* persons.

The equal sharing context is powerful in realizing several instructional goals. It introduces a motivation for equal partitioning of a whole, which is lacking in the purely part-whole approach. It is useful in communicating the idea that a fraction denotes a quantity and a sense of how large the quantity is. Comparison tasks set in sharing contexts elicit a variety of strategies from children and support their reasoning about fraction magnitude. Similarly sharing contexts are helpful in understanding and generating equivalent fractions. The sharing context can be extended to make sense of the addition and subtraction of fractions. Many of these aspects have been explored and implemented in fraction instruction in the last few decades, including by researchers in India (for example, see Subramanian & Verma, 2009). One important issue to address is to lead students from interpreting the fraction as composed of two different quantities to understanding it as denoting a single quantity. For example, 2/3 is not only 2 rotis for 3 people, but the quantity of roti in one share (2/3 of one roti). One way of dealing with this is to connect the share and the measure interpretation together in an explicit manner (see Naik & Subramaniam, 2008). Figure 4 shows an example of a student's work of this kind.

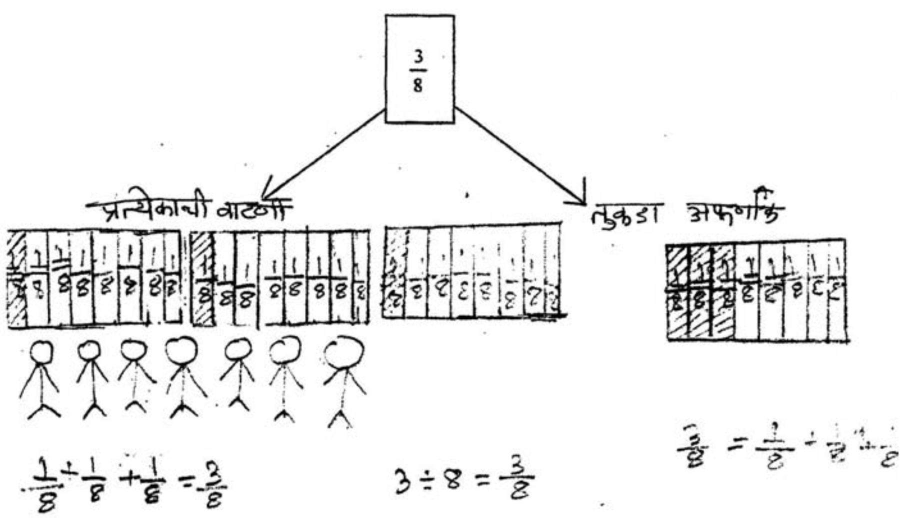


Figure 4: **Connecting the quotient (left) and the part-whole (right) interpretations of the fraction 3/8.**

Overview of the Research on Rational Number Learning from the Mid-70s to the Mid-90s

I have summarized what appear from the present standpoint to be some of the more important contributions of this period of research. It should be mentioned that although this period saw intensive research, its impact on curricula and actual teaching was limited, perhaps because the research findings could not be easily integrated to yield a coherent picture. Reviewing the research of the period, Hunting, Davis and Pearn (1992) drew the rather pessimistic conclusion that "no real progress was being made"(quoted in Lamon, 2007, p. 646). Another reason for the lack of impact on actual instruction may have been the distance of the research studies from actual classroom settings. Behr et al. (1992) remarked that teaching intervention studies on the topic of fractions were limited. Streefland's (1993) work that we have referred to above was an exception in making instructional approach the central focus of research, and in adopting the teaching experiment as the basic methodology. In the 1990s and later, teaching experiments became more numerous.

In the perception of one of the leading researchers (Lamon, 2007), the period after the mid-90s saw relatively less attention to the topic domain of rational numbers on the part of researchers. Although these remarks seem pessimistic about the impact of the research done in the 80s and 90s, and about the intensity of work done subsequently, this may need to be qualified. Lamon herself has authored a book, *Teaching Fractions, Ratio and Proportion*, which illuminates the core initial fraction ideas and the connections between fractions and proportional reasoning in detail (Lamon, 1996, 2nd edition, 2005). The book, which includes many examples of students' reasoning and a large repertoire of activities for students and teachers, exemplifies pedagogical content knowledge in this topic domain and is useful to teachers and curriculum designers. The work of Streefland and his colleagues at the Freudenthal Institute discussed above has had an impact in making the sharing context central to many approaches to teaching fractions. van Galen et al. (2008) draw on this and other work to formulate teaching-learning trajectories for the topic of fractions, proportion and percentages.

Much of the work done in the decades of the 1990s and 2000s extended the themes of the earlier phase of research. The sub-construct theory illuminates students' difficulties with fractions, points to the need for moving beyond the part-whole interpretation of fractions and provides a guideline for instructional design. Hence many researchers have assumed that fractions have multiple interpretations in different contexts and have based their studies and discussions on this assumption. Views about the efficacy of simultaneously teaching several interpretations of fractions diverge. While some studies have claimed efficacy for teaching multiple interpretations simultaneously (Moseley, 2005), others have argued that one interpretation must be dealt in depth so that students can reason confidently on the basis of that interpretation (Lamon, 2007). Some researchers have attempted to directly test the explanatory power of the sub-construct theory, but have not succeeded in establishing clear support for all the sub-constructs identified by the theory (Charalambous & Pitta-Pantazi, 2007). The equal sharing context engages children and elicits multiple reasoning strategies. Empson et al. (2005) developed an elaborate classification of such strategies in a characterization of the anticipatory multiplicative reasoning involved in equal sharing tasks. Teaching experiments such as those by Streefland (1993) and others have used fraction interpretations other than the part-whole interpretation in developing powerful instructional approaches. In the approach adopted by Maher and colleagues, the measure construct was foregrounded and students had an opportunity to flexibly change units and target measures while working with a set of Cuisinaire rods (Steencken & Maher, 2003).

An important strand of research that emerged from the semantic analysis of rational numbers was the study of children's fraction schemes. As discussed above, Lesh and his colleagues analysed multiplicative reasoning in terms of manipulations on composite units. This provided a framework to analyse children's responses and to study the development of multiplicative thinking in the context of their adapting responses. An extended research programme undertaken by Leslie Steffe aimed at describing children's action schemes involving composite units (Steffe, 1992; Steffe & Olive, 2010). A central aim of the research programme was to describe the evolving schemes underlying multiplicative thinking as a modification and development of children's whole number schemes. The theme of the relation between whole number and fraction understanding is important and we shall return to it.

Following the influential work of Liping Ma (1999) on the mathematical knowledge of elementary school teachers from the U.S. and China, many researchers became interested in the teaching and learning of fractions, and in teachers' knowledge of fractions in Asian countries. The most telling differences in the knowledge of teachers from the U.S. and from China in Ma's study were found in the topic of division of fractions. Several subsequent studies have explored teachers' knowledge of this topic in Asian countries (Li & Huang, 2008). Other studies have compared how the topic of fractions is treated in textbooks from different countries (Alajmi, 2011).

A group of researchers from Greece have attempted to study the learning of fractions from the standpoint of conceptual change, an approach that has been successful in the research on children's understanding of science (Vamvakoussi & Vosniadou, 2004). In the conceptual change framework, difficulty in learning a new concept arises from the fact that it is in conflict with a robust conceptual structure or theory that is already in place. Children frequently respond to this conflict by accommodating the new concept, or new data, within the framework of the old concept, leading in many cases to 'synthetic' or 'hybrid' conceptions. Indeed for children who have been learning whole numbers over a period of a few years, fractions present new rules and relationships, which conflict with the whole number framework. Stafylidou and Vosniadou (2004) present a list of important elements of this conflict which include differences in symbolization, ordering, the nature of the unit and the procedures for operating with fractions. In their study, they attempted to explain students' erroneous responses as originating in correct notions about natural numbers, that were extended incorrectly to the rational numbers. This is a confirmation of what has been recognized for long among researchers, namely, that children's whole number conceptions can be hurdles in learning fractions (see for example Streefland, 1993). Vamvakoussi and Vosniadou (2004) argue that one of the fundamental conceptual changes necessary in going over from natural numbers to rational numbers is recognition of the property of density. Rational numbers are dense in the sense that there are infinitely many rational numbers between any two rational numbers. In contrast, natural numbers are discrete in the sense that there are no natural numbers between two consecutive natural numbers. In a study designed to test this hypothesis, they found that 9th graders had the most difficulty in internalizing the property of density and had less difficulty in identifying algebraic properties of rational and real numbers such as the existence of additive and multiplicative inverses.

The conceptual change approach, together with approaches that emphasize the discontinuity between whole number and rational number understanding has been criticized recently by Siegler, Thompson and Schneider (2011). They propose an alternative theory of numerical development that emphasizes the continuity in children's growing understanding of number. The understanding of the magnitude of a number, whether a whole number or a rational number, is the key idea that reflects children's understanding of a number domain. Siegler et al. take numerical development as "coming to understand that all real numbers have magnitudes that can be ordered and assigned specific locations on number lines" (p. 274). In their study, they found that ability to estimate the magnitude and the place of a given number on a numberline was a key correlate of children's proficiency in fraction arithmetic as well as of overall mathematical knowledge. Judgements about the magnitude of a fraction involved strategic reasoning, and children invoked a multiplicity of strategies to make these decisions. Some of the incorrect strategies were due to drawing inaccurate analogies with strategies for whole numbers. Rather than interpret this as a discontinuity between whole number and rational number understanding, they point out that drawing incorrect analogies on the basis of appealing surface similarity rather than structural similarity, is a common challenge in learning across several domains. The idea that fractions like whole numbers have magnitudes and can be ordered on the number line may precisely be the key connecting idea that may help students master the domain of rational numbers. This theory is promising in terms of implicit suggestions for what must be emphasised in the teaching of fractions.

Agendas for Research on Fractions and Multiplicative Reasoning

As discussed in the previous sections, a vast number of research studies have been undertaken in the last three and a half decades on the teaching and learning of fractions in school and related topics such as proportional and multiplicative thinking. This body of research has moved beyond the analysis of student errors and difficulties in implementing fraction computation algorithms. It has shown how fraction interpretations are varied depending on the context of application and that this diversity is not explicitly addressed in the typical school curriculum. It has pointed to important underlying concepts and action schemes such as equipartitioning, unit composition and flexible unitization. It has explored the pedagogical possibilities of rich contexts such as equal sharing. It has uncovered students' spontaneous ways of reasoning in contexts while generating, comparing or ordering fractions. It has illuminated the conceptual changes that children need to make in extending their whole number knowledge to the domain of fractions.

Although the research on children's learning of fractions is vast, it has had limited impact on curriculum design and on the teaching and learning of fractions. For example, we do not find the widespread use of sharing contexts to support students in making sense of fractions or to elicit reasoning about the magnitude of fractions. Although many textbooks have introduced interpretations of fractions other than part-whole, such as the measure or the operator interpretation, the interpretations tend to be disparate. There is no clear vision of a fully developed concept of fraction that connects and integrates the different interpretations into a coherent whole. A part of the reason is the complexity of the topic of fractions and the connections between fractions and other topics.

It is interesting to compare this situation with the topic domain of whole numbers, which is arguably a simpler domain. Research on whole number learning has led to a more coherent picture of how students' knowledge in the domain evolves and to a larger impact on curricula. A recent attempt to explicitly articulate such a picture uses the construct of a *learning-teaching trajectory* as a cohering frame to integrate research findings in the domain (Sarama & Clements, 2009). Learning trajectories are frequently described as conjectured progressions of learning experiences that students encounter as they move from informal to complex, refined and powerful ideas over time.

The idea of learning trajectories has re-emerged as a useful way of organizing and disseminating complex research findings about student learning in specific topic domains. The recent usage of the phrase in the mathematics education research literature can be traced to Simon (1995), for whom 'hypothetical learning trajectory' was a construct that a teacher might use to integrate her learning goals for the students with the students' own mathematical thinking. The phrase was intended to stress the openness of the teacher towards students' ways of thinking and the need to adjust one's teaching in accordance with them, a basic tenet of the constructivist approach to mathematics teaching and learning. As indicated above, the phrase has acquired a broader meaning in current research literature as a construct that can serve to integrate research findings about teaching and learning in a topic domain (Confrey, Maloney, Nguyen, Mojica & Myers, 2009).

Learning trajectories in specific areas have also been viewed as connecting research with curriculum design and teaching practice, as bridges that connect 'grand theories' in education with specific theories and instructional practice (Sarama & Clements, 2009). If it were possible to construct coherent learning trajectories for major curricular strands, it would have obvious advantages. Among other benefits, knowledge of learning trajectories would strengthen domain based pedagogical content knowledge leading to effective teaching. The identification of general learning paths is possible only when there are commonalities in the learning progression of individual children under diverse conditions of learning. Such commonality could result from the existence of general mental structures that underlie mathematical competence with a common developmental path or the existence of shared cultural elements that shape learning and instruction. These factors hold to a large extent for the domain of whole numbers and this is reflected in the learning trajectories that have been described for the domain of whole number learning by different researchers (Fuson, 2009; Sarama & Clements, 2009).

While the idea of a learning trajectory has been productive in the topic domain of whole number addition and subtraction, it has been difficult to extend it to the topic domain of multiplicative thinking. Researchers at the Freudenthal Institute, having previously published a description of learning–teaching trajectories for the whole number domain, published a learning–teaching trajectory for Fractions, percentages, decimals and proportions in 2005 (English version published as van Galen et al., 2008). Here the authors present learning–teaching trajectories separately for the four topic areas of proportions, fractions, percentages and decimals, while acknowledging their inter-connectedness. An attempt is made at the end of the presentation to lay down general learning attainment targets common to these topics. These include important ideas and their interconnections, how knowledge of numerical relationships can be used mindfully in solving problems, and examples of reasoning. An examination of the description of the learning-teaching trajectories and a comparison with themes discussed in the literature reveals that ideas and topics included represent a small selection of what is important in the domain. Indeed, the book is considerably smaller than the corresponding book for whole numbers although the topic domain itself is significantly larger.

An ambitious attempt to synthesize a large body of research on rational number using learning trajectories as a frame has recently been attempted by Jere Confrey and her colleagues (Confrey et al., 2009; Confrey & Maloney, 2010). Confrey and Maloney (2010) identify seven interconnected learning trajectories as forming the topic domain of rational numbers: equipartitioning, ratio, division/multiplication, similarity/scaling, area/volume, decimals/percents and fraction as number. Of these, they provide a description of the trajectory for equipartitioning in terms of a two-dimensional matrix with proficiency levels along one dimension and task classes along the other. The 'task' dimension arises from the fact that tasks requiring equipartitioning may be of various levels of difficulty. This is illustrated in the manner in which the strategy of 'composition of factors' is used: when a student asked to divide a rectangular whole into eight equal parts, she may begin by splitting the rectangle into four equal parts with vertical cuts and then halve the parts with a horizontal cut. This strategy is easier to implement for 'eight' equal parts (a 2^n split), than for 'six' equal parts (partitioning into 3 equal parts is an odd-split, which is harder). Confrey et al.'s learning trajectory matrix for equipartitioning is an elaborate 16 × 13 matrix. While this is a fairly exhaustive description of the many ways in which children achieve equipartitioning of a whole, it is not clear if instruction should be designed to follow the progression in such a matrix.

From these efforts, one can see that an attempt to organize research findings on the learning of fractions and multiplicative reasoning into a coherent picture capable of informing instruction design has begun. At present however, one cannot still discern themes around which research findings can cohere and reflect a cumulative trend. I will end this review by suggesting three themes of research that are important. One of these themes is well researched, while the other two need to be addressed more centrally.

The three broad themes that must form the core strands around which research findings can be integrated to yield possible learning trajectories are children's thinking, cultural supports for learning and acquiring symbolic facility. The first of these themes is about children's intuitive thinking, about their spontaneous or untutored responses to tasks. Instruction consistent with a constructivist approach would give central place to such responses as starting points for instruction. Two constructs which try to capture children's thinking, emerging from slightly different research perspectives are *strategies* and *action schemes*. The construct of 'strategy' is closer to the descriptive level and groups responses to tasks in terms of patterns that are observed frequently. A large number of studies attempt to uncover and classify children's strategies in multiplicative reasoning tasks. Some approaches aim at a developmental account centred on strategies, exemplified by the microgenetic studies by Siegler and his colleagues of children's addition strategies (Shrager & Siegler, 1998). Action schemes go further than strategies in level of theorising, and are mental constructs or objects that are posited as the source of children's thinking and response patterns. Many researchers, influenced by Piaget and Post-Piagetian work such as the semantic analyses of rational number constructs, have attempted to use schemes as organizing constructs to describe children's thinking in the rational number domain and its development. This has led to an understanding of a variety of schemes important for multiplicative thinking such as equipartitioning, unit iteration, unitizing and many-many correspondence. An elaborate account of the emergence of children's fractions schemes from whole number schemes is presented by Steffe and Olive (2010). This theme is perhaps the most developed in the extant literature on the learning of fractions and multiplicative reasoning.

A second theme of research that is needed to identify learning trajectories centres around the sources and support for learning in the culture and seeks to illuminate the relation between out-of-school mathematics and school mathematics. With regard to fractions, everyday experience exposes children to a few basic fractions, which may have special words in local languages (Subramaniam & Naik, 2010). The more significant form of support that everyday experience provides comes from the diversity of contexts in which fraction words are used, reflecting a richness of fraction interpretations in everyday usage. Many everyday situations in which people deal with quantities involve proportional relations and call for multiplicative thinking. Students, especially those participating in household income generation are likely to be familiar with such contexts. Given the ubiquity of proportional relations, and therefore the importance of multiplicative thinking, such studies can contribute to both identifying general principles and developing localized versions of learning trajectories. Another domain that is rich in terms of out-of-school mathematics is measurement. Informal work contexts incorporate many measurement modes and units, often expressed in simple fractions, all of which are potentially rich starting points for instruction (Subramaniam & Bose, 2012). Studies that uncover out-of-school knowledge and look for opportunities to connect it with school mathematics may confront complex issues because the culture that students are a part of is varied depending on location and social stratum and also changing (Subramaniam, 2012). Such studies are also relevant to broader issues of equity and the relation of mathematics education to empowering individuals.

A third theme of research that is important is how children bridge spontaneous ways of thinking and symbolic routines. As children learn to solve more complex problems they must increasingly rely on the mathematical power made available through symbolization. Yet, they must make sense of the symbols and their transformations, drawing on their intuitive understanding of situations and on previous symbolic knowledge. In the domain of multiplicative reasoning, the primary symbolic tools consist of the fraction notation and the arithmetic of fractions. These symbolic tools consolidate and extend the ability to represent and manipulate multiplicative relations. They provide the tools to deal with the full range of situations involving proportionality and also prepare the student for algebra. This theme of research, which is relatively less developed, has fruitful connections with the research in learning algebra – an intensively researched domain in mathematics education. Empson, Levi and Carpenter (2011) have recently explored the use of relational thinking in the learning of fraction arithmetic, where relational thinking involves children's use of fundamental properties of operations and equality to develop efficient solution strategies. Focus on relational thinking has been shown to be important in early algebra instruction (Fujii & Stephens, 2008).

Research on the learning of fractions and multiplicative reasoning is a mature field with a large number of research studies done over several decades. However the impact of this research on curriculum and instructional design is proportionately small. There is a need to move to research programs that accumulate robust findings in a manner that can provide explicit guidelines for instruction, which in turn can be tested by intervention studies. Two relatively under-researched themes may be important to move to this phase: the study of cultural supports for learning fractions and multiplicative reasoning and the acquisition of symbolic capability. Research interest in these questions is active and one anticipates that studies addressing these issues will increase. This would contribute to consolidating the impact of findings in this domain.

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DISCUSSION

Chair- Lynn Webb, Nelson Mandela Metropolitan University, Port Elizabeth, South Africa

Q1: Multiplication has one reference of changing operation while the other being the additive one. I am also reminded that this was one of the mistakes which Aristotle himself made while computing and trying to answer the problem of the law of free fall. Keeping this in mind why is it not emphasized in the curriculum that these are two distinct operations and also corresponding to that there are two inverse operations? There might be other kinds of multiplicative thinking that one needs to look at. The question is why is it not clearly mentioned in the textbook? What is the reason, why mathematicians want to keep it as a single operation?

KS: That's absolutely right. First of all it ought to be there in the curriculum very clearly. Whenever we have done work with the teachers this is always something of an eye opener. There's a paradox where you prove that one rupee is equal to one paisa, as in if you put 100 as 10 into 10 and then you say 1/10 into 1/10 and so on. Actually, this is a good way to open the idea when you are doing the multiplication operation, so something funny is happening. The units are changing and this should be a part of the curriculum and common knowledge of teachers which should be used in their teaching in a very central way. Regarding why is it not there, that's something I can speculate about. I think it is a certain attitude to mathematics. Its also a disconnect between mathematics and the way mathematics is used in the world. There are many reasons which I can speculate about. There is no hard and fast reason. Its just tradition and we need to change it.

Q2: My question is whether it is possible to develop a conceptual understanding of multiplicative thinking, fractions and algebra without numbers. My reason for asking this is because in some indigenous parts of Australia, for numbers and also some elements, I have found that in early childhood, among very young children the concept of measurement and comparison is far more natural than concept of counting.

KS: It's a good question and thank you for raising this question. There are clearly two points here. Measurement originates in completely different structures and counting has a different origin. So you can have both of these built up little by little. There have been curricular experiments starting with the measurement idea rather than counting idea. It has been tried out in Russia and also elsewhere. I don't have a position. It can be that one goes some distance building the idea of numbers but even in place like Papua New Guinea or New Zealand all of these are entering into economy which are characterized by currency, by decimal structures. It's becoming part of the knowledge of the culture, very rapidly in many places. If you look at Geoffrey Saxe's work, when he visited Papua New Guinea again, he found that there is integration of all these cultures into the monetary economy. So it's really changing and something that, I would say, counting is really the starting point and it's the foundation, and one can go very far ahead. It might be that these things have independent origins but they are very tightly interconnected and one must build these interconnections. I don't have a clear position on where to start. I prefer to start from counting.